334 Homework 10

Daniel Rui - 12/5/19

Problem §4.2 - 2

Let $f : [a, b] \to \mathbb{R}$ be an integrable function.

(a) We want to show that the graph of f in $\mathbb{R}^2$ has zero content. (Hint: Given partition of $[a, b]$, interpret $Spfspf$ as sum of areas of rectangles that cover the graph of f)

Given any partition P, f is contained within the rectangles whose heights are upper bounded by M (the supremum on the set) and are lower bounded by m. In other words,

$$\{(x, y) : y = f(x), a \leq x \leq b\} \subseteq \bigcup_{I \in P} I \times [m_I, M_I]$$

Now the content of this set is simply the sum of the contents of the individual rectangles (because the intersections between the rectangles are line segments with content zero), which are all $\ell(I) \cdot (M_I - m_I)$ where $\ell(I)$ is the length of the interval (which we know to be $b - a$ for an interval $[a, b]$). We see that

$$\sum_{I \in P} \ell(I) \cdot (M_I - m_I) = \sum_{I \in P} \ell(I) M_I - \sum_{I \in P} \ell(I) m_I = S_P f - s_P f$$

but because f is integrable, for every $\epsilon > 0$, we can find $S_P f - s_P f < \epsilon$, the graph of f must have zero content.

(b) Suppose $f \geq 0$ and let $S = \{(x, y) : x \in [a, b], 0 < y < f(x)\}$. We want to show that S is measurable and that its area (as defined in this section) equals $\int_a^b f(x)\, dx$.

Let M be the supremum of f on the entire set $[a, b]$. Then consider any partition P of $[a, b]$, and take a corresponding partition Q of $[0, M]$, s.t. for every $I \in P$, M_I and m_I are both in Q. Then,

$$S \subseteq \bigcup_{I \in P} [0, y_1] \cup \ldots \cup [y_j, M_I] = \bigcup_{I \in P} [0, M_I] \quad \text{and} \quad S \supset \bigcup_{I \in P} [0, y_1] \cup \ldots \cup [y_k, m_I - \epsilon] = \bigcup_{I \in P} [0, m_I - \epsilon]$$

or more explicitly,

$$\text{content of} \bigcup_{I \in P} [0, m_I - \epsilon] \leq \text{inner area of } S \leq \text{outer area of } S \leq \text{content of} \bigcup_{I \in P} [0, M_I]$$

The contents of $\bigcup_{I \in P} [0, M_I]$ and $\bigcup_{I \in P} [0, m_I - \epsilon]$ respectively are $S_P f$ and $s_P f - \epsilon \cdot (b - a)$, and because ϵ can be chosen arbitrarily small, and because the difference between $S_P f$ and $s_P f$ gets arbitrarily small, S is measurable and has content equal exactly to $\int_a^b f\, dx$.

Problem 3

Let S be bounded set in $\mathbb{R}^2$. Show that S and S^{int} have the same inner area. (Hint: For any rectangle contained in S, there are slightly smaller rectangles contained in S^{int}).

Let S be contained in some rectangle $[m_x, M_x]\times[m_y, M_y]$. We know that $A \subseteq B \implies A^{\text{int}} \subseteq B^{\text{int}}$, and so for every rectangle $R \subseteq S$, we know that $R^{\text{int}} \subset S^{\text{int}}$. Now if $R = [a,b]\times[c,d]$, then $R^{\text{int}}(a,b)\times(c,d)$, so for any $\delta > 0$, consider $R' = [a+\delta/2, b-\delta/2]\times[c+\delta/2, d-\delta/2] \subset (a,b)\times(c,d)$. So the area of R minus the area of R' is $\delta((b-a)+(d-c))-\delta^2$. Thus, for any partition of $[m_x, M_x]\times[m_y, M_y]$ into rectangles, we have a set of rectangles $\mathcal{R}$ s.t. $\bigcup_{R\in\mathcal{R}} R \subseteq S$ and $\bigcup_{R\in\mathcal{R}} R' \subseteq S^{\text{int}}$.

Thus, the difference in areas between $\bigcup_{R\in\mathcal{R}} R$ and $\bigcup_{R\in\mathcal{R}} R'$ is simply

$$\sum_{R\in\mathcal{R}} \delta((b_R - a_R) + (d_R - c_R)) - \delta^2$$

but we can make the observation that for any rectangle R, $b_R - a_R \leq M_x - m_x$ and similarly $d_R - c_R \leq M_y - m_y$. Thus the above sum is less than or equal to

$$\sum_{R\in\mathcal{R}} \delta(M_x - m_x) + \delta(M_y - m_y) - \delta^2 = \delta(M_x - m_x)|\mathcal{R}| + \delta(M_y - m_y)|\mathcal{R}| - \delta^2|\mathcal{R}|$$

For any fixed $\mathcal{R}$, we can always find $\delta_\epsilon > 0$ s.t. the above expression is less than $\epsilon > 0$ for any ϵ we choose. In other words, we have that

$$\text{area of} \bigcup_{R\in\mathcal{R}} R - \text{area of} \bigcup_{R\in\mathcal{R}} R'_{\delta_\epsilon} < \epsilon/2 \implies \text{area of} \bigcup_{R\in\mathcal{R}} R - \epsilon/2 < \text{area of} \bigcup_{R\in\mathcal{R}} R'$$

and so

$$\text{area of} \bigcup_{R\in\mathcal{R}} R - \epsilon/2 < \text{area of} \bigcup_{R\in\mathcal{R}} R'_{\delta_\epsilon} \leq \text{inner area of } S^{\text{int}} \leq \text{inner area of } S$$

Now for this fixed $\epsilon > 0$, we can always find some partition of rectangles $\mathcal{R}$ s.t.

$$\left|\text{area of} \bigcup_{R\in\mathcal{R}} R - \text{inner area of } S\right| < \epsilon/2 \implies \text{area of} \bigcup_{R\in\mathcal{R}} R > \text{inner area of } S - \epsilon/2$$

and so combining the two inequalities we have that

$$\text{inner area of } S - \epsilon < \text{area of} \bigcup_{R\in\mathcal{R}} R - \epsilon/2 \leq \text{inner area of } S^{\text{int}} \leq \text{inner area of } S$$

and because ϵ is arbitrary, the inner areas of S^{int} and S must be the same.

Problem 4

Let S be bounded set in $\mathbb{R}^2$. Show that S and $\bar{S}$ have the same outer area.

Let m_x and M_x be the infimum and supremum of the x values in the set S and similarly define m_y, M_y. Then define a rectangle $R := [m_x - 1, M_x + 1] \times [m_y - 1, M_y + 1]$ s.t. $S \subseteq \bar{S} \subset R$ (with an additional padding of 1 on all sides). The outer area of $\bar{S}$ plus the inner area of $R \setminus \bar{S}$ is thus the inner area of R. Similarly, the outer area of S plus the inner area of $R \setminus S$ is also the inner area of R. But we have that

$$(R \setminus \bar{S})^{\text{int}} = R^{\text{int}} \setminus \bar{S} = (R \setminus S)^{\text{int}}$$

And because a set and its interior have the same inner area (from Problem 3), both the outer areas of S and $\bar{S}$ are equal to the inner area of R minus the inner area of $R \setminus S$.

Problem 7

Suppose f is continuous on $[a, b]$ and φ is of class C^1 and increasing on $[a, b]$. Show that there is point $c \in [a, b]$ such that

$$\int_a^b f(x)\varphi(x)\,dx = \varphi(a)\int_a^c f(x)\,dx + \varphi(b)\int_c^b f(x)\,dx$$

(Hint: First suppose $\varphi(b) = 0$. Set $F(x) = \int_a^x f(t)\,dt$, integrate by parts to show that $\int_a^b f(x)\varphi(x)\,dx = -\int_a^b F(x)\varphi'(x)\,dx$, and apply Theorem 4.24 to the latter integral. To remove the condition $\varphi(b) = 0$, show that if the conclusion is true for f and φ, it is true for f and $\varphi + C$ for any constant C.)

Following the hint and using integration by parts, we see that

$$\begin{aligned}\int_a^b f(x)\varphi(x)\,dx &= F(x)\varphi(x)\Big|_a^b - \int_a^b F(x)\varphi'(x)\,dx \\ &= F(b)\cdot 0 - 0\cdot\varphi(a) - \int_a^b F(x)\varphi'(x)\,dx = -\int_a^b F(x)\varphi'(x)\,dx\end{aligned}$$

Because φ has a continuous derivative and is increasing on $[a, b]$, $\varphi'(x) \geq 0$ and so the mean value theorem says that there is some c s.t.

$$-\int_a^b F(x)\varphi'(x)\,dx = -F(c)\int_a^b \varphi'(x)\,dx = -F(c)\cdot(\varphi(b) - \varphi(a)) = F(c)\varphi(a)$$

or in other words

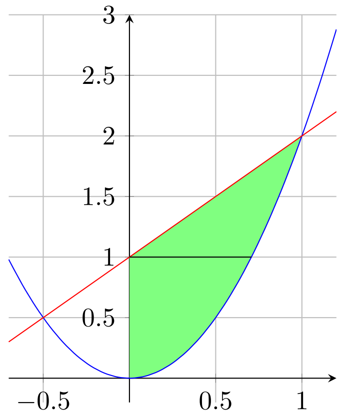
$$\int_a^b f(x)\varphi(x)\,dx = F(c)\varphi(a) = \varphi(a)\int_a^c f(x)\,dx$$

And thus for any arbitrary C^1 increasing function $\phi(x)$ s.t. $\phi(b) = B$, take $\varphi(x) = \phi(x) - B$ s.t. $\varphi(b) = 0$ (and also of course $\varphi(x)$ still C^1 and increasing), we have that

$$\begin{aligned}\int_a^b f(x)\phi(x)\,dx = \int_a^b f(x)(\varphi(x) + B)\,dx &= \int_a^b f(x)\varphi(x)\,dx + \int_a^b Bf(x)\,dx \\ &= \varphi(a)\int_a^c f(x)\,dx + B\int_a^c f(x)\,dx + B\int_c^b f(x)\,dx \\ &= (\varphi(a) + B)\int_a^c f(x)\,dx + B\int_c^b f(x)\,dx = \phi(a)\int_a^c f(x)\,dx + \phi(b)\int_c^b f(x)\,dx\end{aligned}$$

Problem §4.3 - 6

Fill in the blanks: $\int_0^1 \int_{2x^2}^{x+1} f(y)\,dy\,dx = \int_0^1 [\qquad]\,dy + \int_1^2 [\qquad]\,dy$. The region in question is



Thus,

$$\begin{aligned}\int_0^1 \int_{2x^2}^{x+1} f(y)\,dy\,dx &= \int_0^1 \int_0^{\sqrt{y/2}} f(y)\,dx\,dy + \int_1^2 \int_{1-y}^{\sqrt{y/2}} f(y)\,dx\,dy \\ &= \int_0^1 f(y)\sqrt{y/2}\,dy + \int_1^2 f(y)(\sqrt{y/2} - (1-y))\,dy\end{aligned}$$

Problem 7

Given continuous function $g : \mathbb{R} \to \mathbb{R}$, let $h(x) = \int_0^x \int_0^y g(t)\,dt\,dy$. That is, h is obtained by integrating g twice, starting the integration at 0. Show that h can be expressed as a single integral, namely, $h(x) = \int_0^x (x-t)g(t)\,dt$.

The region in question is the upper left half triangle of the $[0,x] \times [0,x]$ square. Because g is continuous, we can use Fubini's theorem, which tells us that

$$\int_0^x \int_0^y g(t)\,dt\,dy = \int_0^x \int_t^x g(t)\,dy\,dt = \int_0^x g(t) \int_t^x 1\,dy\,dt = \int_0^x g(t)(x-t)\,dt$$

Problem 13

Let $f(x,y) = y^{-2}$ if $0 < x < y < 1$, $f(x,y) = -x^{-2}$ if $0 < y < x < 1$, and $f(x,y) = 0$ otherwise, and let S be the unit square $[0,1] \times [0,1]$

(a) Show that f is not integrable on S, but that $f(x,y)$ is integrable on $[0,1]$ as function of x for each fixed y and as function of y for each fixed x.

We know that f is not integrable on S because f is unbounded (take for example a sequence of points $(1/n, 2/n)$, yielding $f(x,y) = \frac{n^2}{4}$), and unbounded functions are not integrable because the upper sums will always be ∞.

If we fix an x, say x_0, then

$$f(x_0, y) = \begin{cases} -\dfrac{1}{x_0^2} & \text{if } 0 < y < x_0 \\ \dfrac{1}{y^2} & \text{if } x_0 < y < 1 \end{cases}$$

which is a bounded function that's only discontinuous at finitely many points, so it's integrable. Similarly,

$$f(x, y_0) = \begin{cases} -\dfrac{1}{x^2} & \text{if } y_0 < x < 1 \\ \dfrac{1}{y_0^2} & \text{if } 0 < x < y_0 \end{cases}$$

We will compute the integrals here first before we move on:

$$\int_0^1 f(x_0, y)\, dy = -\frac{1}{x_0^2} \cdot x_0 + \int_{x_0}^1 \frac{1}{y^2}\, dy = -\frac{1}{x_0} - 1 + \frac{1}{x_0} = -1$$

and

$$\int_0^1 f(x, y_0)\, dx = \frac{1}{y_0^2} \cdot y_0 + \int_{y_0}^1 -\frac{1}{x^2}\, dx = \frac{1}{y_0} + 1 - \frac{1}{y_0} = 1$$

(b) Show by explicit calculation that the iterated integrals $\int_0^1 \int_0^1 f(x,y)\, dx\, dy$ and $\int_0^1 \int_0^1 f(x,y)\, dy\, dx$ both exist but are unequal. From above, we see that

$$\int_0^1 \int_0^1 f(x,y)\, dx\, dy = \int_0^1 1\, dy = 1 \quad \text{and} \quad \int_0^1 \int_0^1 f(x,y)\, dy\, dx = \int_0^1 -1\, dx = -1$$

Problem §4.4 - 7

Find the mass of a ball of radius R if the mass density is c times the distance from the boundary of the ball. From this description, $\rho(x, y, z) = c(R - \sqrt{x^2 + y^2 + z^2})$. This sounds like a job for spherical coordinates: $x = r \sin\varphi \cos\theta, y = r \sin\varphi \sin\theta, z = r\cos\varphi$ (a topic already discussed and developed in the book). Thus,

$$\begin{aligned} \iiint_S \rho\, dV &= \int_0^\pi \int_0^{2\pi} \int_0^R c(R - r)(r^2 \sin\varphi)\, dr\, d\theta\, d\varphi \\ &= c \int_0^\pi \int_0^{2\pi} \frac{1}{3} R^4 \sin\varphi - \frac{1}{4} R^4 \sin\varphi\, d\theta\, d\varphi \\ &= 2c\pi \left(\frac{1}{12} R^4 \right) \int_0^\pi \sin\varphi\, d\varphi = 4c\pi \left(\frac{1}{12} R^4 \right) = \frac{1}{3} c\pi R^4 \end{aligned}$$

Problem 13

Let S be the region in the first quadrant bounded by the curves $xy = 1, xy = 3$ and $x^2 - y^2 = 1, x^2 - y^2 = 4$. Compute $\int\int_S (x^2 + y^2)\, dA$. Let $u = xy$, $v = x^2 - y^2$, and $\mathbf{G}(x, y) = (xy, x^2 - y^2)$. Shuffling some variables around and applying the quadratic formula (and noticing that $\sqrt{v^2 + 4u^2}$ is never less than v), we have that

$$\mathbf{G}^{-1}(u,v) = \left(\sqrt{\frac{v+\sqrt{v^2+4u^2}}{2}}, \sqrt{\frac{-v+\sqrt{v^2+4u^2}}{2}}\right)$$

keeping in mind that $1 \le u \le 3$, $1 \le v \le 4$, and $x, y \ge 0$. Because $\mathbf{G}$ has an inverse, it is bijective; furthermore it's C^1 because we are adding and multiplying C^1 functions. Then

$$|\det D\mathbf{G}(x,y)| = \left|\det \begin{pmatrix} y & x \\ 2x & -2y \end{pmatrix}\right| = |-2y^2 - 2x^2| = 2(x^2+y^2)$$

and so

$$\iint_S \frac{1}{2} \cdot 2(x^2+y^2)\, dx\, dy = \int_1^3 \int_1^4 \frac{1}{2}\, dv\, du = 3$$

Problem 14

Use the transformation $x = u - uv, y = uv$ to evaluate $\iint_S (x+y)^{-1}\, dA$ where S is the region in the first quadrant between the lines $x+y=1$ and $x+y=4$. From the equations involving x, y, u, v, we know that $u = x+y$ and $v = y/(x+y)$, and that $1 \le u \le 4$. Along the y-axis $x=0$, $v=1$ and along the x-zxis $y=0$, $v=0$ (and everywhere strictly inside the first quadrant $0 < v < 1$, so v ranges from 0 to 1. In other words, we have

$$\mathbf{G}(u,v) = (u - uv, uv) \quad \text{and} \quad \mathbf{G}^{-1}(x,y) = \left(x+y, \frac{y}{x+y}\right)$$

and so we see that because $\mathbf{G}$ has an inverse, it is bijective; furthermore it's C^1 because we are adding, multiplying, and dividing by C^1 functions (dividing by things that aren't 0). Observe that

$$|\det D\mathbf{G}(u,v)| = \left|\det \begin{pmatrix} 1-v & -u \\ v & u \end{pmatrix}\right| = |u(1-v)+uv| = |u| = u$$

and so

$$\iint_S \frac{1}{x+y}\, dA = \int_0^1 \int_1^4 \frac{1}{u} u\, du\, dv = 3$$

Problem 15

Use "double polar coordinates" $x = r\cos\theta, y = r\sin\theta, z = s\cos\varphi, w = s\sin\varphi$ in $\mathbb{R}^4$ to compute the 4-dimensional volume of the sphere S: $x^2+y^2+z^2+w^2 = R^2$. Notice that excluding $(r,s) = (0,0)$, i.e. $(x,y,z,w) = (0,0,0,0)$, we have

$$\mathbf{G}(r,s,\theta,\varphi) = (r\cos\theta, r\sin\theta, s\cos\varphi, s\sin\varphi)$$

and

$$\mathbf{G}^{-1}(x,y,z,w) = \left(\sqrt{x^2+y^2}, \sqrt{z^2+w^2}, \begin{cases} \operatorname{sgn}(y)\cdot\frac{\pi}{2} & x=0 \\ \arctan\left(\frac{y}{x}\right) & x_{>0}, y_{\geq 0} \\ \arctan\left(\frac{y}{x}\right)+\pi & x<0 \\ \arctan\left(\frac{y}{x}\right)+2\pi & x_{>0}, y_{<0} \end{cases}, \begin{cases} \operatorname{sgn}(w)\cdot\frac{\pi}{2} & z=0 \\ \arctan\left(\frac{w}{z}\right) & z_{>0}, w_{\geq 0} \\ \arctan\left(\frac{w}{z}\right)+\pi & z<0 \\ \arctan\left(\frac{w}{z}\right)+2\pi & z_{>0}, w_{<0} \end{cases} \right)$$

and so we see that because $\mathbf{G}$ has an inverse, it is bijective; furthermore it's C^1 because multiplying by C^1 functions. Observe that

$$\begin{aligned} |\det D\mathbf{G}(r,s,\theta,\varphi)| &= \left| \det \begin{pmatrix} \cos\theta & 0 & -r\sin\theta & 0 \\ \sin\theta & 0 & r\cos\theta & 0 \\ 0 & \cos\varphi & 0 & -s\sin\varphi \\ 0 & \sin\varphi & 0 & s\cos\varphi \end{pmatrix} \right| \\ &= \left| \cos\theta\Big(-r\cos\theta(s\cos^2\varphi + s\sin^2\varphi)\Big) - r\sin\theta\Big(\sin\theta(s\cos^2\varphi+s\sin^2\varphi)\Big) \right| \\ &= |-rs| = rs \end{aligned}$$

and so

$$\begin{aligned} \iiiint_S 1\,dw\,dz\,dy\,dx &= \int_0^{2\pi}\int_0^{2\pi}\int_0^R\int_0^{\sqrt{R^2-r^2}} rs\,ds\,dr\,d\theta\,d\varphi \\ &= \int_0^{2\pi}\int_0^{2\pi}\int_0^R \frac{1}{2}r(R^2-r^2)\,dr\,d\theta\,d\varphi \\ &= \int_0^{2\pi}\int_0^{2\pi} \frac{1}{4}R^4 - \frac{1}{8}R^4\,d\theta\,d\varphi = 2\pi\cdot 2\pi\frac{1}{8}R^4 = \frac{\pi^2}{2}R^4 \end{aligned}$$

Problem §5.1 - 2

Express the arc length of the following curves in terms of the integral

$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2 t}\,dt \quad (0<k<1)$$

for suitable values of k.

(a) An ellipse with semimajor axis a and semiminor axis b can be parameterized as $x(\theta) = a\cos\theta, y(\theta) = b\sin\theta$. The entire arclength is simply four times the arclength between $0 \leq \theta \leq \frac{\pi}{2}$, so using the formula for arclength we get that

$$\begin{aligned} L &= 4\int_0^{\pi/2} |(-a\sin\theta, b\cos\theta)|\,d\theta = 4\int_0^{\pi/2} \sqrt{(-a\sin\theta)^2+(b\cos\theta)^2}\,d\theta \\ &= 4\int_0^{\pi/2}\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}\,d\theta = 4\int_0^{\pi/2}\sqrt{(a^2-b^2)\sin^2\theta+b^2\sin^2\theta+b^2\cos^2\theta\,d\theta} \\ &= 4b\int_0^{\pi/2}\sqrt{\frac{a^2-b^2}{b^2}\sin^2\theta+1} = 4bE\left(\sqrt{\frac{b^2-a^2}{b^2}}\right) \end{aligned}$$

(b) The portion of the intersection of the sphere $x^2 + y^2 + z^2 = 4$ and the cylinder $x^2 + y^2 - 2y = 0$ lying in the first octant can be parameterized as follows: the intersection of the cylinder and the xy-plane is $x^2+y^2-2y = 0$ (but interpreted in 2-dimensions), which is just $x^2+(y-1)^2 = 1$, which can be parameterized as $x(\theta) = \cos\theta, y(\theta) = 1+\sin\theta$ where $\theta \in [-\pi/2, \pi/2]$ (to keep it in the first octant). Since we are only considering the first octant, we know $z = \sqrt{4 - x^2 - y^2}$, and so we can use the previous parameterizations to see that $z(\theta) = \sqrt{4 - \cos^2\theta - (1 + \sin\theta)^2} = \sqrt{2 - 2\sin\theta}$. Thus, using the formula we get that the arclength is

$$L = \int_{-\pi/2}^{\pi/2} \left|\left(-\sin\theta, \cos\theta, \frac{-\cos\theta}{\sqrt{2-2\sin\theta}}\right)\right| d\theta = \int_{-\pi/2}^{\pi/2} \sqrt{1 + \frac{\cos^2\theta}{2(1-\sin\theta)}}\, d\theta$$

We are reminded of the identity $2\sin^2\phi = 1 - \cos 2\phi$, so the variable change $2\phi = (\pi/2 - \theta)$ we get that

$$\begin{aligned} L &= \int_{\pi/2}^{0} \sqrt{1 + \frac{\sin^2(2\phi)}{2(1-\cos 2\phi)}}(-2)\, d\phi = 2\int_0^{\pi/2} \sqrt{1 + \frac{2^2\sin^2\phi\cos^2\phi}{2\cdot 2\sin^2\phi}}\, d\phi \\ &= 2\int_0^{\pi/2} \sqrt{1+\cos^2\phi}\, d\phi = \frac{1}{2}\int_0^{\pi/2} \sqrt{2-\sin^2\phi}\, d\phi \\ &= 2\int_0^{\pi/2} \sqrt{2}\sqrt{1 - \frac{1}{2}\sin^2\phi}\, d\phi = 2\sqrt{2}E\left(\frac{1}{\sqrt{2}}\right) \end{aligned}$$

Problem 7

Let $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous map, and C let be C^1 curve in $\mathbb{R}^n$. Let $\mathbf{g}(t)$, $a \le t \le b$ parameterize C.

(a) We want to prove that that $|\int_C \mathbf{F}\, ds| \le \int_C |\mathbf{F}|\, ds$. From the definition, we know that $\int_C f\, ds = \int_a^b f(\mathbf{g}(t))|\mathbf{g}'(t)|\, dt$, and Proposition 5.8 says that $|\int_a^b \mathbf{F}(t)\, dt| \le \int_a^b |\mathbf{F}(t)|\, dt$, and thus

$$\left|\int_C \mathbf{F}\, ds\right| = \left|\int_a^b \mathbf{F}(\mathbf{g}(t))|\mathbf{g}'(t)|\, dt\right| \le \int_a^b \Big|\mathbf{F}(\mathbf{g}(t))|\mathbf{g}'(t)|\Big|\, dt = \int_a^b |\mathbf{F}(\mathbf{g}(t))| \cdot |\mathbf{g}'(t)|\, dt = \int_C |\mathbf{F}|\, ds$$

(b) In the case $m = n$, show that $|\int_C \mathbf{F}\, d\mathbf{x}| \le \int_C |\mathbf{F}|\, ds$. Recall that by definition $\int_C \mathbf{F}\, d\mathbf{x} = \int_a^b \mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}(t)\, dt$, and Cauchy-Schwarz: $|\langle \mathbf{u}, \mathbf{v}\rangle| \le |\mathbf{u}||\mathbf{v}|$.

$$\left|\int_C \mathbf{F}\, d\mathbf{x}\right| = \left|\int_a^b \mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t)\, dt\right| \le \int_a^b \Big|\mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t)\Big|\, dt = \int_a^b |\mathbf{F}(\mathbf{g}(t))| \cdot |\mathbf{g}'(t)|\, dt = \int_C |\mathbf{F}|\, ds$$

334 Homework 9

Daniel Rui - 11/21/19

Problem §3.4 - 5

Consider the region bounded by $x^2 \leq y \leq 2x^2$ and $1 \leq xy \leq 3$. We can rephrase the first bound to be $1 \leq \frac{y}{x^2} \leq 2$. Now consider the function $\mathbf{f} : (x, y) \mapsto (u, v) := (\frac{y}{x^2}, xy)$, which takes the region described to a rectangle in (u, v)-coordinates.

$$D\mathbf{f} = \begin{pmatrix} \nabla u \\ \nabla v \end{pmatrix} = \begin{pmatrix} -\frac{2y}{x^3} & \frac{1}{x^2} \\ y & x \end{pmatrix} \quad \text{and} \quad \det(D\mathbf{f}) = -\frac{2y}{x^2} - \frac{y}{x^2} = -\frac{3y}{x^2}$$

We can find the inverse mapping $\mathbf{f}^{-1}$ taking $(u, v) \mapsto (x, y)$ by noting that

$$uv^2 = y^3 \implies y = \sqrt[3]{uv^2} \quad \text{and} \quad \frac{v}{u} = x^3 \implies x = \sqrt[3]{\frac{v}{u}}$$

The values in the Frechet derivative are continuous (on the desired region), so $\mathbf{f} \in C^1$. Because the map $\mathbf{f}$ and the map $\mathbf{f}^{-1}$ both exist and are well-defined for the regions in question, both must be 1-to-1: they must be injective because the other is a well-defined function, and they must be surjective because the inverse exists at the desired point.

Problem §4.1 - 1

We have the function

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

We now prove that for any (non-singleton) interval I, $\sup_I f(x) = 1$ and $\inf_I f(x)$. Because the rationals and irrationals are dense (as I proved in Homework 1), every non-singleton interval has a rational and an irrational, and hence f takes on 0 and 1 on every interval. Because f only ever takes on 0 or 1, the supremum on I is 1 and the infimum is 0.

Now for any arbitrary interval $[a, b]$ where $a < b$, any partition we make (of non-singleton intervals of course) will yield upper and lower sums

$$S_P f = \sum_{i=1}^{n} \left(\sup_{[x_i, x_{i+1}]} f \right) \cdot (x_{i+1} - x_i) = \sum_{i=1}^{n} (x_{i+1} - x_i) = b - a$$

while

$$s_P f = \sum_{i=1}^{n} \left(\inf_{[x_i, x_{i+1}]} f \right) \cdot (x_{i+1} - x_i) = \sum_{i=1}^{n} 0 \cdot (x_{i+1} - x_i) = 0$$

Clearly, these results do not depend on the arbitrary partition, and for non-singleton intervals $[a, b]$, the upper and lower sums do not agree (for any partition) and thus the integral does not exist.

Problem 5

First we prove the following lemma: if f, g are both integrable on $[a,b]$ and $f(x) \leq g(x)$, then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

This is rather easy; we know that for any given partition P, $s_P f \leq s_P g$ and $S_P f \leq S_P g$. From integrability, we know that for every $\epsilon > 0$, we can find a partition P s.t.

$$s_P f \leq \int_a^b f(x)\,dx < s_P f + \epsilon \quad \text{and} \quad s_P g \leq \int_a^b g(x)\,dx < s_P g + \epsilon$$

Now suppose that $\int_a^b f(x)\,dx > \int_a^b g(x)\,dx$. Then $\int_a^b f(x)\,dx - \int_a^b g(x)\,dx > 0$, so we can take it to be ϵ. Furthermore, from the above inequality we can subtract ϵ from both sides, yielding

$$\int_a^b f(x)\,dx < s_P f + \epsilon \implies \int_a^b f(x)\,dx - \epsilon < s_P f \leq \int_a^b f(x)\,dx < s_P f + \epsilon$$

But from the way we've defined ϵ, $\int_a^b f(x)\,dx - \epsilon = \int_a^b g(x)\,dx$! In other words, we've just shown that for this particular partition P, we have that

$$s_P g \leq \int_a^b g(x)\,dx < s_P f \leq \int_a^b f(x)\,dx \leq s_P f + \epsilon$$

or more clearly, that $s_P g < s_P f$ — contradiction.

Now to prove the actual result: if f is integrable on $[a,b]$, $|f|$ is also integrable on $[a,b]$. Let M be the supremum of f on some interval I and let m to be the infimum of f on the same interval I. Then, let A ("A" for absolute value) be the supremum of $|f|$ on some interval I and let a to be the infimum of $|f|$ on the same interval I.

If $M, m \geq 0$, then $M = A$ and $m = a$, so $M - m = A - a$. If $M, m \leq 0$, then $A = -m$ and $a = -M$, so $M - m = -a - (-A) = A - a$. Finally, if $M > 0$ and $m < 0$, then $A = \max\{M, -m\}$ and $a = \min\{M, -m\} > 0$. Keep in mind that if $\alpha = \max\{\beta, \gamma\}$ were $\beta, \gamma > 0$, then α is definitely less than the sum $\beta + \gamma$. Thus, we know that $A - a < A < M + (-m) = M - m$. Thus for any partition, we've just proved here that any one term of $S_P|f| - s_P|f|$ is less than or equal to the corresponding term of $S_P f - s_P f$.

Thus, if for particular partition P, $S_P f - s_P f < \epsilon$, then also $S_P|f| - s_P|f| < \epsilon$, which means $|f|$ is integrable over $[a,b]$. Finally, because $f \leq |f|$ AND $-f \leq |f|$, by the lemma above,

$$\int_a^b f\,dx \leq \int_a^b |f|\,dx \quad \text{and} \quad -\int_a^b f\,dx \leq \int_a^b |f|\,dx \implies \left|\int_a^b f\,dx\right| \leq \int_a^b |f|\,dx$$

Problem 6

We have a convergent sequence $\{x_1, x_2, \ldots\}$. Let us call the limit L. Convergence tells us that given any positive number, for example $\epsilon/4$, there is some $N \in \mathbb{N}$ s.t. for all $n \geq N$, $|x_n - L| < \epsilon/4$, or in other words, $x_n \in (L - \frac{\epsilon}{4}, L + \frac{\epsilon}{4})$. Denote this interval I_N, and take $I_1, \ldots, I_{N-1}$ to simply be the intervals $(x_1 - \frac{\epsilon}{2N}, x_1 + \frac{\epsilon}{2N}) \ldots, (x_{N-1} - \frac{\epsilon}{2N}, x_{N-1} + \frac{\epsilon}{2N})\}$. Thus, we've just found N intervals, whose total length is $2\epsilon/4 + (N-1) \cdot \frac{\epsilon}{2N} = \epsilon - \frac{\epsilon}{2N} < \epsilon$, and because this is true for any $\epsilon > 0$, the set $\{x_1, x_2, \ldots\}$ has zero content.

Problem 7

We are given that there is a point $x_0 \in [a, b]$ at which f is continuous and positive. By continuity, we know that in particular if we choose $\epsilon = f(x_0)/2$, there is δ such that $|x - x_0| < \delta \implies |f(x) - f(x_0)| < f(x_0)/2 \implies f(x) > f(x_0)/2 > 0$. This tells us that on the interval $(x_0 - \delta, x_0 + \delta)$, $f(x) > f(x_0)/2 > 0$. Now take a partition P of $[a, b]$ s.t. $[x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}]$ is an interval in the partition. Then, we have that (denoting $\ell(I)$ to be the length of the interval):

$$s_P f = \sum_{I \in P} \left(\inf_I f\right) \cdot \ell(I) \geq \left(\inf_{[x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}]} f\right) \cdot \ell(\left[x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}\right]) > \frac{f(x_0)}{2}\delta > 0$$

And because f is integrable on $[a, b]$, we have that for any partition (and thus in particular for our partition above),

$$s_P f \leq \int_a^b f(x)\, dx$$

and so we know that

$$\int_a^b f(x)\, dx > \frac{f(x_0)}{2}\delta > 0$$

Problem 8c

We want to show that for any $c \in \mathbb{R}$,

$$\int_a^b f(x)\, dx = \int_{a-c}^{b-c} f(x+c)\, dx$$

Consider any partition of $[a-c, b-c]$, $P' = \{[x_0 - c, x_1 - c], \ldots, [x_{n-1} - c, x_n - c]\}$. Now consider an individual term of $s_{P'} f(x+c)$:

$$\inf_{x \in [x_i - c, x_{i+1} - c]} f(x+c) \cdot (x_{i+1} - c - (x_i - c)) = \inf_{x \in [x_i - c, x_{i+1} - c]} f(x+c) \cdot (x_{i+1} - x_i)$$

Because we know $x \in [x_i - c, x_{i+1} - c]$, $x + c$ must be in $[x_i, x_{i+1}]$, and so the above is equivalent to taking the infimum of f taking values from $[x_i, x_{i+1}]$. In other words, the above term is exactly equal to

$$\inf_{[x_i, x_{i+1}]} f(x)(x_{i+1} - x_i)$$

Thus,

$$\begin{aligned}
s_{P'}f(x+c) &= \sum_{i=0}^{n-1} \inf_{x\in[x_i-c,x_{i+1}-c]} f(x+c)\cdot(x_{i+1}-c-(x_i-c)) \\
&= \sum_{i=0}^{n-1} \inf_{[x_i,x_{i+1}]} f(x)(x_{i+1}-x_i) = s_P f(x)
\end{aligned}$$

where $P = \{[x_0, x_1], \ldots, [x_{n-1}, x_n]\}$ and $s_P f(x)$ is the lower sum of $f(x)$ on P. Similarly, the corresponding upper sum $S_{P'}f(x+c)$ is exactly equivalent to $S_P f(x)$. These results hold for any partition P' of $[a-c, b-c]$ and corresponding shifted partition P.

We want to prove that $f(x+c)$ is integrable on $[a-c, b-c]$: from the integrability of $f(x)$ on $[a, b]$, we know that for every $\epsilon > 0$, there exists some partition P s.t. $S_P f(x) - s_P f(x)\epsilon$, but we can always shift P to P' (by subtracting c from each of the endpoints of its intervals). From our work above, we know that for any P' and corresponding shifted partition P, $S_{P'}f(x+c) - s_{P'}f(x+c) = S_P f(x) - s_P f(x)$, so we now have that $S_{P'}f(x+c) - s_{P'}f(x+c) < \epsilon$ (again, for arbitrary ϵ) which means that $f(x+c)$ is integrable on $[a-c, b-c]$.

From the integrability of f on $[a, b]$, we know that $s_P f(x) \to \int_a^b f(x)\,dx$. From the integrability of $f(x+c)$ on $[a-c, b-c]$, we know that $s_{P'}f(x+c) \to \int_{a-c}^{b-c} f(x+c)\,dx$. But $s_{P'}f(x+c) = s_P f(x)$, and the same exact sequence must converge to the same exact value; therefore,

$$\int_{a-c}^{b-c} f(x+c)\,dx = \int_a^b f(x)\,dx.$$

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Daniel Rui - 11/14/19

Problem §3.2 - 4

$$\varphi(s) = \begin{cases} s^2 & \text{if } s \geq 0 \\ -s^2 & \text{if } s < 0 \end{cases}$$

(a) $\varphi'(s) = \begin{cases} 2s & \text{if } s > 0 \\ -2s & \text{if } s < 0 \end{cases}$ by differentiability rules. At $s = 0$,

$$0 \leq \left| \frac{\varphi(s)}{s} \right| \leq \frac{s^2}{|s|} = |s|$$

As $s \to 0$, $|s| \to 0$, and thus by the squeeze theorem, $\lim_{s \to 0} \left| \frac{\varphi(s)}{s} \right|$ exists and equals 0. Thus, $\lim_{s \to 0} \frac{\varphi(s)}{s}$ exists and equals 0. This is the derivative of $\varphi(s)$ at 0, so $\varphi'(0)$ exists and equals 0. Thus, combining everything we know into one formula we see that $\varphi'(s) = 2|s|$, which is continuous. Hence $\varphi \in C^1$.

(b) Let $f(t) = (\varphi(\cos t), \varphi(\sin t))$. If $t \in [0, \frac{\pi}{2}]$, both cos and sin are ≥ 0, so $x(t) = \cos^2 t$ and $y(t) = \sin^2 t$ and thus $x + y = 1$, and because we $x, y \geq 0$ we are only considering the line $x + y = 1$ in the first quadrant (and the axes). Similar analysis reveals that $t \in [\frac{\pi}{2}, \pi]$ corresponds to the portion of the line $y - x = 1$ in the second quadrant (and the axes); $t \in [\pi, \frac{3\pi}{2}]$ corresponds to the portion of the line $-y - x = 1$ in the third quadrant (and the axes); and $t \in [\frac{3\pi}{2}, 2\pi]$ corresponds to the portion of the line $-y + x = 1$ in the fourth quadrant (and the axes). This forms a square with corners on $(\pm 1, 0), (0, \pm 1)$. For $t \geq 2\pi$ and $t \leq 0$, by the periodicity of sin and cos, $f(t)$ just traces these four line segments over and over again.

Furthermore, we have that $f'(t) = \big(-(\sin t)(2|\cos t|), (\cos t)(2|\sin t|) \big)$ which equals $(0, 0)$ at $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, i.e. $t = \frac{n}{2}\pi$ for $n \in \mathbb{Z}$, which are the corners of the square.

Problem 6

We have $F_1, F_2 \in C^1$ on some open set U in the plane, and $F_3 = F_1 F_2$. Let $S_i = \{x \in U : F_i(x) = 0\}$

(a) We want to show that $S_3 = S_1 \cup S_2$. Obviously, if $x \in S_1 \cup S_2$, then at least one of $F_1, F_2 = 0$, and obviously something times 0 is 0, and so $x \in S_3$. Now if $x \in S_3$, x must be in at least one of S_1, S_2, because suppose not: then, the product of two non-zero numbers is 0, which is impossible. Thus $S_3 = S_1 \cup S_2$.

(b) From the chain rule,

$$\partial_x F_3 = \partial_x F_1 F_2 + F_1 \partial_x F_2 \quad \text{and} \quad \partial_y F_3 = \partial_y F_1 F_2 + F_1 \partial_y F_2$$

so

$$\nabla F_3 = \nabla F_1 F_2 + F_1 \nabla F_2$$

so of course if $a \in S_1 \cap S_2$ meaning $F_1(a) = F_2(a) = 0$, then $\nabla F_3(a) = 0$.

Problem §3.2 - 4

(a) The planes $x - 2y + z = 3$ and $2x - y - z = -1$ are perpendicular to $(1, -2, 1)$ and $(2, -1, -1)$ respectively. Thus the line of the intersection is perpendicular to both these vectors, so we can find it via the cross product. $(1, -2, 1) \times (2, -1, -1) = (3, 3, 3)$, and one point on both the planes is $(-\frac{5}{3}, -\frac{7}{3}, 0)$ and so a parameterization of the line is $l(t) = (-\frac{5}{3}, -\frac{7}{3}, 0) + (3, 3, 3)t$.

(b) The planes $x + 2y = 3$ and $y - 3z = 2$ are perpendicular to $(1, 2, 0)$ and $(0, 1, -3)$ respectively. Thus the line of the intersection is perpendicular to both these vectors, so we can find it via the cross product. $(1, 2, 0) \times (0, 1, -3) = (-6, 3, 1)$, and one point on both the planes is $(-1, 2, 0)$ and so a parameterization of the line is $l(t) = (-1, 2, 0) + (-6, 3, 1)t$.

Problem 5

(a) The intersection between $x^2 + y^2 + z^2 = 1$ and $x + z = 1$ is simply the result of substitution $z = 1 - x$ into the first equation:

$$\begin{aligned}
x^2 + y^2 + (1-x)^2 = 1 &\implies x^2 + y^2 + x^2 - 2x + 1 = 1 \implies 2x^2 - 2x + y^2 = 0 \\
&\implies 2x^2 - 2x + \frac{1}{2} + y^2 = \frac{1}{2} \implies 2\left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{2} \\
&\implies \frac{\left(x - \frac{1}{2}\right)^2}{1/4} + \frac{y^2}{1/2} = 1
\end{aligned}$$

This is an ellipse with center $(1/2, 0)$ and $a = \frac{1}{2}$, $b = \frac{\sqrt{2}}{2}$, which we know can be parameterized as

$$x(\theta) = \frac{1}{2} + \frac{1}{2}\cos\theta \quad \text{and} \quad y(t) = \frac{\sqrt{2}}{2}\sin\theta \quad \text{and} \quad z(\theta) = \frac{1}{2} - \frac{1}{2}\cos\theta$$

where $\theta \in [0, 2\pi)$

(b) The point in question, $(\frac{1}{2}, -\frac{\sqrt{2}}{2}, \frac{1}{2})$ is attained at $\theta = -\frac{\pi}{2}$. Thus the tangent vector at the point is just the derivative evaluated at the point, i.e.

$$\left(-\frac{1}{2}\sin\theta, \frac{\sqrt{2}}{2}\cos\theta, \frac{1}{2}\sin\theta\right) \text{ evaluated at } \theta = \frac{\pi}{2} \text{ is } \left(\frac{1}{2}, 0, \frac{1}{2}\right)$$

and so the parameterization of the tangent line at the point is

$$l(t) = \left(\frac{1}{2}, -\frac{\sqrt{2}}{2}, \frac{1}{2}\right) + \left(\frac{1}{2}, 0, \frac{1}{2}\right)t$$

334 Homework 7

Daniel Rui - 11/7/19

Problem §2.8 - 1c

We want to find the critical points of $f(x, y) = (x - 1)(x^2 - y^2) = x^3 - xy^2 - x^2 + y^2$. The gradient of f is $\nabla f(x, y) = \langle 3x^2 - y^2 - 2x, -2xy + 2y \rangle$ and the Hessian is

$$H(x, y) = \begin{pmatrix} 6x - 2 & -2y \\ -2y & -2x + 2 \end{pmatrix}$$

The critical points are where $\nabla f(x, y) = 0$, i.e. the solutions to the equations $2y(1 - x) = 0$ and $3x^2 - y^2 - 2x = 0$. To satisfy the first equation, there are two cases: $x = 1$ and $y = 0$. For $x = 1$, $3 - y^2 - 2 = 0 \implies y = \pm 1$. For $y = 0$, $3x^2 - 2x = 0 \implies x = 0, x = \frac{2}{3}$. In total, we have four points: $(0, 0), (\frac{2}{3}, 0), (1, 1)$, and $(1, -1)$. Evaluating the Hessian at each of these points, we get

$$H(0,0) = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}, \quad H(\tfrac{2}{3}, 0) = \begin{pmatrix} 2 & 0 \\ 0 & \frac{2}{3} \end{pmatrix}, \quad H(1,1) = \begin{pmatrix} 4 & -2 \\ -2 & 0 \end{pmatrix}, \quad H(1,-1) = \begin{pmatrix} 4 & 2 \\ 2 & 0 \end{pmatrix}$$

and the corresponding determinants are

$$\det(H(0,0)) = -4, \quad \det(H(\tfrac{2}{3}, 0)) = \frac{4}{3}, \quad \det(H(1,1)) = -4, \quad \det(H(1,-1)) = -4$$

Finally, note that $H(\frac{2}{3}, 0)$ is positive definite (because $f_{xx}(\frac{2}{3}, 0) = 2 > 0$ and $\det(H(\frac{2}{3}, 0)) = \frac{4}{3} > 0$), so the critical points are, respectively, a saddle, a minimum, a saddle, and another saddle.

Problem 4

We are given a function $f(x, y) = (y - x^2)(y - 2x^2) = y^2 - 3x^2y + 2x^4$. The gradient is $\nabla f(x, y) = (8x^3 - 6xy, 2y - 3x^2)$ and the Hessian is

$$H(x, y) = \begin{pmatrix} -6y + 24x^2 & -6x \\ -6x & 2 \end{pmatrix}$$

(a) At the origin, $\nabla f(0, 0) = 0$ and $\det(H(0, 0)) = 0$, so by definition the origin is a degenerate critical point.

(b) We want to show that the function $g(t) = f(at, bt)$ (for any (a, b) except $(0, 0)$) has a local minimum at the origin, but that f does not have a local minimum at the origin.

Writing out $g(t)$ explicitly, we get $g(t) = b^2t^2 - 3a^2bt^3 + 2a^4t^4$, and $g'(t) = 2b^2t - 9a^2bt^2 + 8a^4t^3$, and $g''(t) = 2b^2 - 18a^2bt + 24a^4t^2$, so $g'(0) = 0$ and $g''(0) = 2b^2$. This tells us that if $b \neq 0$, then g has a minimum at 0. Now if $b = 0$, then $g(t) = 2a^4t^4$ which obviously has a minimum at $t = 0$ (where $a \neq 0$ because $(a, b) \neq (0, 0)$).

However, consider the restriction of f on the curve $(x, y) = (t, \frac{3}{2}t^2)$. Then $g(t) = f(t, \frac{3}{2}t^2) = \frac{1}{2}t^2(-\frac{1}{2}t^2) = -\frac{1}{4}t^2$, which has a MAXIMUM at $t = 0$ (i.e. at the origin). Thus, $f(x, y)$ can not have a local minimum at the origin, which corroborates our observation above that the origin is a degenerate critical point.

Problem 5

Let H be the Hessian of f. We want to show that for any unit vector $\mathbf{u}$, $H\mathbf{u}\cdot\mathbf{u}$ is the second directional derivative of f in the direction $\mathbf{u}$.
By definition, (taking gradients to be row vectors, and $\mathbf{u}$ to be a column vector)

$$H\mathbf{u}\cdot\mathbf{u} = \begin{bmatrix} [\nabla(\partial_1 f)] \\ \vdots \\ [\nabla(\partial_n f)] \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} [\nabla(\partial_1 f)]\mathbf{u} \\ \vdots \\ [\nabla(\partial_n f)]\mathbf{u} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \sum_{i=1}^{n} u_i[\nabla(\partial_i f)]\mathbf{u} = \sum_{i=1}^{n} u_i \sum_{j=1}^{n} (\partial_{j,i} f) u_j$$

The first directional derivative is $[\nabla f]\mathbf{u} = \sum_{i=1}^{n} u_i(\partial_i f)$ (keeping in mind we are making gradients row vectors), and so the second directional derivative is

$$\begin{aligned} \nabla\left(\sum i = 1^n u_i(\partial_i f)\right)\mathbf{u} &= \left[\sum_{i=1}^{n} u_i \nabla(\partial_i f)\right]\mathbf{u} = \left[\sum_{i=1}^{n} u_i \left\langle(\partial_{1,i} f), \ldots, (\partial_{n,i}) f\right\rangle\right]\mathbf{u} \\ &= \left\langle \sum_{i=1}^{n} u_i(\partial_{1,i} f), \ldots, \sum i = 1^n u_i(\partial_{n,i}) f \right\rangle \mathbf{u} \\ &= \sum_{j=1}^{n} u_j \sum_{i=1}^{n} u_i(\partial_{j,i} f) \end{aligned}$$

By equality of mixed partials, the two double sums are the same.

Problem §2.9 - 13

The line between $(1, 0, 0)$ and $(0, 1, 0)$ can be parameterized as $f(t_1) = (1 - t_1, t_1, 0)$, and the line between $(0, 0, 0)$ and $(1, 1, 1)$ can be parameterized as $g(t_2) = (t_2, t_2, t_2)$. We want to minimize the function $d(t_1, t_2) = |f(t_1) - g(t_2)|$. For ease, we will minimize d^2, which yields the same results as minimizing d. Writing things out, we have that $d^2(t_1, t_2) = (1 - t_1 - t_2)^2 + (t_1 - t_2)^2 + (-t_2)^2 = 1 - 2t_1 + 2t_1^2 - 2t_2 + 3t_2^2$. The gradient is therefore $\nabla d^2(t_1, t_2) = \langle 4t_1 - 2, 6t_2 - 2\rangle$ and the Hessian is

$$H(t_1, t_2) = \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix}$$

so the only critical point is $(\frac{1}{2}, \frac{1}{3})$ is a minimum. Thus, the lines are closest when the point on f is $(\frac{1}{2}, \frac{1}{2}, 0)$ and the point on g is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Problem 15

We have two planes $x + z = 4$ and $3x - y = 6$ that intersect in a line L. We want to find the minimum distance L is from the origin; that is, we want to minimize $d^2 = x^2 + y^2 + z^2$ where (x, y, z) have to satisfy the equations of the two planes above. We do this using the method of Lagrange multipliers; we want to find the points where $\nabla d^2(x, y, z) = \lambda_1 \nabla(x+z) + \lambda_2 \nabla(3x - y)$, or: $2x = \lambda_1 + 3\lambda_2$, $2y = -\lambda_2$, and $2z = \lambda_1$.

We can write $2x = \lambda_1 + 3\lambda_2 = 2z - 6y$, or more simply $x = z - 3y \implies z = x + 3y$. Substituting this into the first of the original constraints we get that $2x + 3y = 4$. With the second original constraint, this yields $11x = 22 \implies x = 2$. Solving for everything else yields that the closest point is $(x, y, z) = (2, 0, 2)$.

Problem 18

We want to maximize the product $x_1 \cdot x_2 \cdots x_n$ constrained by $x_1 + \ldots + x_n = c$ where all the $x_i \geq 0$. Clearly, if any of the $x_i = 0$, the product will be 0, which is definitely a minimum. Thus, let us consider $x_i > 0$. Using Lagrange multipliers, we see that the critical points occur when

$$\begin{bmatrix} x_2 \cdots x_n \\ \vdots \\ x_1 \cdots x_{n-1} \end{bmatrix} = \begin{bmatrix} \lambda \\ \vdots \\ \lambda \end{bmatrix} \implies \begin{bmatrix} x_1 \cdots x_n \\ \vdots \\ x_1 \cdots x_n \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{bmatrix} \implies x_1 = \ldots = x_n$$

where $\lambda \neq 0$ because all the $x_i \neq 0$. Now the fact that $x_1 = \ldots = x_n$ means that each of the $x_i = \frac{c}{n}$, and so the product is $(\frac{c}{n})^n$. Thus, the n-th root of the maximum value of the product is equal to $\frac{c}{n}$, which is exactly the arithmetic mean. That means, if we know that the sum of the x_i is c, then the n-th root of the product of the x_i is less than $\frac{c}{n}$, where equality holds when all the x_i are equal.

Problem §3.1 - 5

Let $F(x, y)$ be a function with continuous derivative (i.e. of class C^1) s.t. $F(0, 0) = 0$. What conditions on F would guarantee that the equation $F(F(x, y), y) = 0$ could be solved for y as a C^1 function of x around $(0, 0)$?

Define $g(x, y) = F(F(x, y), y)$. By the implicit function theorem, we want to find the conditions on which $\partial_y g(x, y) \neq 0$. By the chain rule, $\partial_y g(x, y) = \partial_x F(F(x, y), y) \cdot \partial_y F(x, y) + \partial_y F(x, y) = \partial_y F(x, y)(\partial_x F(F(x, y), y) + 1)$. This equals 0 at $(0, 0)$ if $\partial_y F(0, 0) = 0$ or $\partial_x F(F(0, 0), 0) = -1 \implies \partial_x F(0, 0) = -1$.

Problem 8

We have the equations $xy^2 + xzu + yv^2 = 3$ and $u^3yz + 2xv - u^2v^2 = 2$. Can we solve for u, v as functions of x, y, z near $(x, y, z, u, v) = (1, 1, 1, 1, 1)$?

Define $F(x, y, z, u, v) = xy^2 + xzu + yv^2 - 3$ and $G(x, y, z, u, v) = u^3yz + 2xv - u^2v^2 - 2$. Then

$$\det\left(\frac{\partial(F, G)}{\partial(u, v)}\right) = \det\begin{pmatrix} xz & 2yv \\ 3u^2yz - 2uv^2 & 2x - 2u^2v \end{pmatrix}$$

so at $(1, 1, 1, 1, 1)$ the determinant is $0-2 = -2$ which is not 0, and so by the implicit function theorem for systems of equations, we can solve for u, v in terms of x, y, z near $(1, 1, 1, 1, 1)$.

Postscript

Goodness, these homeworks just keep getting more and more cursed — as the number of dimensions increases, the level of rigor just goes through the floor (looking at you, Lagrange multipliers). May this deluge of horror come to an end soon!

334 Homework 6

Daniel Rui - 10/31/19

Problem §2.5 - 4

If $u = x^2 + 3y^2$, then $\frac{\partial u}{\partial x} = 2x$. If we further say that $y = xz$, then $u = x^2 + 3x^2z^2$, so $\frac{\partial u}{\partial x} = 2x + 6xz^2$

Problem 6

We have a function F of three variables (x, y, z). Consider the set of points s.t. $F(x, y, z) = 0$. For the point $\mathbf{x} = (x_0, y_0, z_0)$ s.t. $F(\mathbf{x}) = 0$, the implicit function theorem gives that we *can* parameterize z as some function of x, y I will denote $f_3(x, y)$. Define $f_1(x, y) = x$ and $f_2(x, y) = y$. Then $F(f_1(x, y), f_2(x, y), f_3(x, y)) = 0 \implies \frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0$, and we have that

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial f_1}\frac{\partial f_1}{\partial x} + \frac{\partial F}{\partial f_2}\frac{\partial f_2}{\partial x} + \frac{\partial F}{\partial f_3}\frac{\partial f_3}{\partial x} = \frac{\partial F}{\partial f_1} + \frac{\partial F}{\partial f_3}\frac{\partial f_3}{\partial x} = 0$$

We will now do what's called a pro-gamer move: f_1 is our original x, so $\frac{\partial F}{\partial f_1} = \frac{\partial F}{\partial x} = F_x$ and similarly $\frac{\partial F}{\partial f_3} = F_z$. Solving for $\frac{\partial f_3}{\partial x}$ now gives $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z}$.

Similarly, if we parameterize $y = f_2(x, z)$ (yes, sorry about bad naming conventions) then we can say that $\dfrac{\partial y}{\partial z} = -\dfrac{F_z}{F_y}$; and if we parameterize $x = f_1(y, z)$, then $\dfrac{\partial x}{\partial y} = -\dfrac{F_y}{F_x}$. Thus, the product of these three is

$$\frac{\partial z}{\partial x}\frac{\partial y}{\partial z}\frac{\partial x}{\partial y} = (-1)^3\frac{F_x}{F_z}\frac{F_z}{F_y}\frac{F_y}{F_x} = -1$$

Problem 7

We have variables E, T, V, P and satisfies $f(E, T, V, P) = 0$ and $g(E, T, V, P) = 0$. Now if we represent $E = \phi(V, T)$ and $P = \varphi(V, T)$ for some functions ϕ and φ, then we are given that $\frac{\partial E}{\partial V} - T\frac{\partial P}{\partial T} + P = 0$. To use our newly defined functions, this is exactly the same as saying that $\frac{\partial \phi}{\partial V} - T\frac{\partial \varphi}{\partial T} + P = 0$

Now if we completely disregard our dependencies up until this point and instead say that $E = \chi(P, T)$ for some function χ and $V = \psi(P, T)$ for some function ψ (i.e. P and T are the "independent variables"). Then $\frac{\partial E}{\partial P} = \frac{\partial \chi}{\partial P}$. But using our old function ϕ, the chain rule gives that $\frac{\partial E}{\partial P} = \frac{\partial \phi}{\partial V}\frac{\partial \psi}{\partial P}$. Also, $1 = \frac{\partial P}{\partial P} = \frac{\partial \varphi}{\partial V}\frac{\partial \psi}{\partial P}$. Rearranging these two most recent equations gives

$$\frac{\partial \phi}{\partial V} = \frac{\frac{\partial E}{\partial P}}{\frac{\partial \psi}{\partial P}} \quad \text{and} \quad \frac{1}{\frac{\partial \psi}{\partial P}} = \frac{\partial \varphi}{\partial V}$$

Also by the chain rule, $\frac{\partial P}{\partial T} = \frac{\partial \varphi}{\partial V}\frac{\partial \psi}{\partial T} + \frac{\partial \varphi}{\partial T}\frac{\partial T}{\partial T} = \frac{\partial \varphi}{\partial V}\frac{\partial \psi}{\partial T} + \frac{\partial \varphi}{\partial T}$. But P and T are "independent" in our current framework, so $\frac{\partial P}{\partial T} = 0$. This means that $-\frac{\partial \varphi}{\partial V}\frac{\partial \psi}{\partial T} = \frac{\partial \varphi}{\partial T}$. Substituting our value of $\frac{\partial \varphi}{\partial V}$ from above, we get

$$-\frac{\frac{\partial \psi}{\partial T}}{\frac{\partial \psi}{\partial P}} = \frac{\partial \varphi}{\partial T}$$

Substituting into our original equality: $\frac{\partial \phi}{\partial V} - T\frac{\partial \varphi}{\partial T} + P = 0$, we get

$$\frac{\frac{\partial E}{\partial P}}{\frac{\partial \psi}{\partial P}} - T\left(-\frac{\frac{\partial \psi}{\partial T}}{\frac{\partial \psi}{\partial P}}\right) + P = 0 \implies \frac{\partial E}{\partial P} + T\frac{\partial \psi}{\partial T} + P\frac{\partial \psi}{\partial P} = 0 \implies \frac{\partial E}{\partial P} + T\frac{\partial V}{\partial T} + P\frac{\partial V}{\partial P} = 0$$

as desired.

Problem §2.6 - 5

We are given a homogeneous function f of degree a on $\mathbb{R}^n$. Denote $\varphi(t) = f(t\mathbf{x}) = t^a f(\mathbf{x})$. Differentiating φ w.r.t. t twice yields $\varphi''(t) = a(a-1)t^{a-2}f(\mathbf{x})$.

On the other hand, $f(t\mathbf{x}) = f(tx_1, \ldots, tx_n)$. By the chain rule,

$$\frac{\partial f}{\partial t}(t\mathbf{x}) = [\partial_1 f](t\mathbf{x})x_1 + \ldots + [\partial_n f](t\mathbf{x})x_n$$

Now for each $[\partial_i f]$, the derivative w.r.t. t is again chain rule:

$$\frac{\partial[\partial_i f]}{\partial t}(t\mathbf{x}) = [\partial_1[\partial_i f]](t\mathbf{x})x_1 + \ldots + [\partial_n[\partial_i f]](t\mathbf{x})x_n = \sum_{j=1}^{n}[\partial_j[\partial_i f]](t\mathbf{x})x_j$$

Thus, the second derivative of $f(t\mathbf{x})$ w.r.t. t will just be

$$\frac{\partial^2 f}{(\partial t)^2}(t\mathbf{x}) = x_1\sum_{j=1}^{n} x_j[\partial_j[\partial_1 f]](t\mathbf{x}) + \ldots + x_n\sum_{j=1}^{n} x_j[\partial_j[\partial_n f]](t\mathbf{x}) = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j[\partial_j[\partial_i f]](t\mathbf{x})$$

Because $\varphi''(t) = \frac{\partial^2 f}{(\partial t)^2}(t\mathbf{x})$, we can use the above equalities evaluated at $t = 1$ to get

$$a(a-1)f(\mathbf{x}) = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j[\partial_j[\partial_i f]](\mathbf{x})$$

Problem 7

We have $u = f(x-ct) + g(x+ct)$. Then $u_{tt} = (-c)^2 f''(x-ct) + c^2 g''(x-ct)$ and $u_x x = f''(x-ct) + g''(x-ct)$ so $u_{tt} = c^2 u_x x$. Note: f and g both seem to be single variable functions, so f' and g' denote just the normal derivative for normal single variable functions.

Problem 8

We are given that $F(\mathbf{x}, t) = \dfrac{g(ct-r)}{r}$ where $r = |\mathbf{x}|$. Preliminary observation: $\dfrac{\partial r}{\partial x} = \dfrac{1}{2} \cdot 2x \cdot \dfrac{1}{r} = \dfrac{x}{r}$.

Then we know that

$$F_x = \frac{r \cdot \left(-\frac{x}{r}\right) \cdot g'(ct-r) - g(ct-r) \cdot \frac{x}{r}}{r^2} = -\frac{xg'(ct-r)}{r^2} - \frac{xg(ct-r)}{r^3}$$

and so

$$\begin{aligned}
F_{xx} &= -\frac{r^2(-x \cdot \frac{x}{r} \cdot g''(ct-r) + g'(ct-r)) - 2r\frac{x}{r} \cdot xg'(ct-r)}{r^4} \\
&\quad - \frac{r^3(-x \cdot \frac{x}{r} \cdot g'(ct-r) + g(ct-r)) - 3r^2\frac{x}{r} \cdot xg(ct-r)}{r^6} \\
&= -\frac{g(ct-r)}{r^3} + \frac{3x^2g(ct-r)}{r^5} + \frac{x^2g'(ct-r)}{r^4} - \frac{g'(ct-r)}{r^2} + \frac{2x^2g'(ct-r)}{r^4} + \frac{x^2g''(ct-r)}{r^3} \\
&= -\frac{g'(ct-r)}{r^2} + \frac{x^2g''(ct-r)}{r^3} - \frac{g(ct-r)}{r^3} + \frac{3x^2g'(ct-r)}{r^4} + \frac{3x^2g(ct-r)}{r^5}
\end{aligned}$$

F_{yy} and F_{zz} are just the above except with y's and z's substituted for x's, by symmetry. Keeping in mind that $x^2 + y^2 + z^2 = r^2$, the sum becomes

$$\begin{aligned}
F_{xx} + F_{yy} + F_{zz} &= -\frac{3g'(ct-r)}{r^2} + \frac{r^2g''(ct-r)}{r^3} - \frac{3g(ct-r)}{r^3} + \frac{3r^2g'(ct-r)}{r^4} + \frac{3r^2g(ct-r)}{r^5} \\
&= -\frac{3g'(ct-r)}{r^2} + \frac{g''(ct-r)}{r} - \frac{3g(ct-r)}{r^3} + \frac{3g'(ct-r)}{r^2} + \frac{3g(ct-r)}{r^3} \\
&= \frac{g''(ct-r)}{r}
\end{aligned}$$

Finally,

$$F_t = c\frac{g'(ct-r)}{r} \implies F_{tt} = c^2\frac{g''(ct-r)}{r}$$

so we have the desired equality:

$$F_{xx} + F_{yy} + F_{zz} = \frac{1}{c^2}F_{tt}$$

Problem 9

We are given a function $F(\mathbf{x}) = f(r)$ where again $r = |\mathbf{x}|$. Remember from Problem 8 that for any $i \in \{1 \dots n\}$, $\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$. Then, we have that

$$F_{x_i} = \frac{x_i}{r}f'(r)$$

so

$$\begin{aligned}
F_{x_ix_i} &= \frac{r\left(x_i \cdot \frac{x_i}{r}f''(r) + f'(r)\right) - x_i \cdot f'(r) \cdot \frac{x_i}{r}}{r^2} \\
&= \frac{f'(r)}{r} + \frac{x_i^2f''(r)}{r^2} - \frac{x_i^2f'(r)}{r^3}
\end{aligned}$$

Keeping in mind that $x_1^2 + ... + x_n^2 = r^2$, we have that

$$
\begin{aligned}
\sum_{i=1}^{n} F_{x_i x_i} &= \sum_{i=1}^{n} \frac{f'(r)}{r} + \frac{x_i^2 f''(r)}{r^2} - \frac{x_i^2 f'(r)}{r^3} \\
&= n\frac{f'(r)}{r} + \frac{r^2 f''(r)}{r^2} - \frac{r^2 f'(r)}{r^3} \\
&= n\frac{f'(r)}{r} + f''(r) - \frac{f'(r)}{r} \\
&= (n-1)\frac{f'(r)}{r} + f''(r)
\end{aligned}
$$

Problem §2.7 - 3

Because $x - \frac{1}{6}x^3$ is the Taylor polynomial approximation to $\sin(x)$ of degree 3,

$$|\sin(x) - (x - \frac{1}{6}x^3)| = |R_{0,4}(x)| \leq \frac{1}{5!}|x|^5 \leq \frac{\frac{\pi^5}{2^5}}{5!} = 0.07969 \leq 0.08$$

because $|\sin^{(5)}(x)| = |\sin(x)| \leq 1$. Similarly, if we take the Taylor approximation of degree 5, we get that the difference between that and $\sin(x)$ is

$$|R_{0,6}| \leq \frac{1}{7!}|x|^7 \leq \frac{\frac{\pi^7}{2^7}}{7!} \leq 0.004682 < 0.01$$

Problem 8

Note: remember that by definition

$$P_{a,k}(h) = \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!} h^k$$

The desired limit is

$$\lim_{h\to 0} \frac{R_{a,k}(h)}{h^k} = \lim_{h\to 0} \frac{f(a+h) - P_{a,k}(h)}{h^k}$$

applying l'Hôpital's rule $k-1$ times, we get that this is equal to

$$
\begin{aligned}
\lim_{h\to 0} \frac{(k!)f^{(k-1)}(a+h) - (k!)f^{(k-1)}(a) - (k!)f^{(k)}(a)h}{(k!)h} &= \lim_{h\to 0} \frac{f^{(k-1)}(a+h) - f^{(k-1)}(a) - f^{(k)}(a)h}{h} \\
&= \lim_{h\to 0} \frac{f^{(k-1)}(a+h) - f^{(k-1)}(a)}{h} - f^{(k)}(a)
\end{aligned}
$$

which is just 0 by definition of $f^{(k)}(a)$.

Problem 9

We know by definition of Taylor approximations and remainders that

$$f(a+h) = f(a) + \frac{f'(a)}{1!}h^1 + \ldots + \frac{f^{(k-1)}(a)}{(k-1)!}h^{k-1} + \frac{f^{(k)}(a)}{k!}h^k + R_{a,k}(h)$$

But we are given that $f'(a) = \ldots = f^{(k-1)}(a) = 0$, so that leaves us with

$$f(a+h) = f(a) + \frac{f^{(k)}(a)}{k!}h^k + R_{a,k}(h)$$

By Corollary 2.60 in the text, $R_{a,k}(h) \to 0$ as $h \to 0$, i.e. for every ϵ we can find h small enough that $|R_{a,k}(h)| < \epsilon$. In particular, we make the following choice for ϵ:

$$|R_{a,k}(h)| \leq \frac{1}{2}\left|\frac{f^{(k)}(a)}{k!}h^k\right|$$

If k is even and $f^{(k)} > 0$, then

$$f(a+h) - f(a) = \frac{f^{(k)}(a)}{k!}h^k + R_{a,k}(h) \geq \frac{1}{2}\frac{f^{(k)}(a)}{k!}h^k > 0$$

which holds for all h sufficiently close to a (i.e. in a neighborhood around a), so f is a local minimum at a. Similarly, if k is even but $f^{(k)} < 0$, then

$$f(a+h) - f(a) = \frac{f^{(k)}(a)}{k!}h^k + R_{a,k}(h) \leq -\frac{1}{2}\left|\frac{f^{(k)}(a)}{k!}h^k\right| < 0$$

so $f(a)$ is a local maximum. If k odd, then h^k flips in sign on the left of a and on the right of a, so for $h > 0$ (on the right) and $f^{(k)}(a) > 0$ we have that

$$f(a+h) - f(a) = \frac{f^{(k)}(a)}{k!}h^k + R_{a,k}(h) \geq \frac{1}{2}\frac{f^{(k)}(a)}{k!}h^k > 0$$

but for $h < 0$ (on the left) we have that

$$f(a+h) - f(a) = \frac{f^{(k)}(a)}{k!}h^k + R_{a,k}(h) \leq -\frac{1}{2}\left|\frac{f^{(k)}(a)}{k!}h^k\right| < 0$$

Similarly if $f^{(k)}(a) < 0$, we can see that $f(a+h) - f(a)$ switches sign on the left and right of a; thus in fact f is not a minimum or maximum at a.

334 Homework 5

Daniel Rui - 10/24/19

Problem §2.2 - 7

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) We want to prove that f is continuous at $(0,0)$. Note that by AM-GM, we have that $\sqrt{x^2y^2} \leq \dfrac{x^2+y^2}{2} \implies 2|xy| \leq x^2+y^2 \implies \dfrac{1}{x^2+y^2} \leq \dfrac{1}{2|xy|}$. Thus,

$$|f(x,y) - f(0,0)| = \left|\frac{x^2y}{x^2+y^2}\right| \leq \frac{|x^2y|}{2|xy|} = \frac{|x|}{2}$$

Because $\frac{|x|}{2}$ goes to 0 as $x \to 0$, $|f(x,y)|$ must too, and that would imply that $f(x,y) \to 0$ as $(x,y) \to (0,0)$. Thus the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ exists, so f is continuous at $(0,0)$.

(b) All the directional derivatives exist: we can paramaterize (x,y) approaching along a line as $(r\cos\theta, r\sin\theta)$ as $r \to 0$. Thus, all directional derivatives are just the following limit $\forall\theta \in [0, 2\pi)$:

$$\lim_{r\to 0} \frac{\frac{r^2\cos^2(\theta) r \sin\theta}{r^2\cos^2\theta + r^2\sin^2\theta}}{r} = \cos^2(\theta)\sin\theta$$

so the directional derivative, approaching along a line with slope $\tan(\theta)$ is $\cos^2(\theta)\sin\theta$.

(c) However, f is not differentiable at $(0,0)$. From part (b), $f_x(0,0) = \cos^2(0)\sin(0) = 0$ and $f_y(0,0) = \cos^2(\frac{\pi}{2})\sin(\frac{\pi}{2}) = 0$. From the book, we know that if f were to be differentiable, the directional derivative in along the vector $\mathbf{v}$ would be $\nabla f(0,0)\cdot\mathbf{v}$. But in our case, $\nabla f(0,0) = (0,0)$, so all the directional derivatives would be 0, which is not what we calculated in part (b).

Problem 8

We have a function f on an open set $S \subset \mathbb{R}^n$ that has bounded partial derivatives $\partial_j f$ on S. We want to prove that it's continuous on S. Consider the point $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in S$, and arbitrarily small vector $\mathbf{h}$ such that $\mathbf{a}+\mathbf{h}$ is still in S. To prove continuity, we want the difference $f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) \to 0$ as $\mathbf{h} \to 0$. We can write

$$\begin{aligned} f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) = f(&\mathbf{a} + (h_1, h_2, h_3, \ldots, h_n)) \\ &- f(\mathbf{a} + (0, h_2, h_3, \ldots, h_n)) + f(\mathbf{a} + (0, h_2, h_3, \ldots, h_n)) \\ &- f(\mathbf{a} + (0, 0, h_3, \ldots, h_n)) + f(\mathbf{a} + (0, 0, h_3, \ldots, h_n)) \\ &- \ldots \\ &- f(\mathbf{a} + (0, 0, 0, \ldots, 0, h_n)) + f(\mathbf{a} + (0, 0, 0, \ldots, 0, h_n)) \\ &- f(\mathbf{a} + (0, 0, 0, \ldots 0)) \end{aligned}$$

For ease of notation, denote $(0, \ldots, 0, h_i, \ldots, h_n)$ as $\mathbf{h}_{[i]}$ (and likewise $\mathbf{a}_{[i]} = (0, \ldots, 0, a_i, \ldots, a_n)$. Then denote $g_i(x) = f(a_1, a_2, \ldots, a_{i-1}, x, a_{i+1} + h_{i+1}, \ldots, a_n + h_n)$. By definition,

$$g_i'(x) = \partial_i f(a_1, \ldots, a_{i-1}, x, a_{i+1} + h_{i+1}, \ldots, a_n + h_n)$$

because $\partial_i f$ exists and is bounded on all of S. Thus, all g are differentiable (hence continuous). Using this new notation, we can write the above expansion of $f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a})$ to be:

$$\begin{aligned}
&= \Big(f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}+\mathbf{h}_{[1]})\Big) + \Big(f(\mathbf{a}+\mathbf{h}_{[1]}) - f(\mathbf{a}+\mathbf{h}_{[2]})\Big) - \ldots + \Big(f(\mathbf{a}+\mathbf{h}_{[n]}) - f(\mathbf{a}+\mathbf{0})\Big) \\
&= \Big(g_1(a_1+h_1) - g_1(a_1)\Big) + \Big(g_2(a_2+h_2) - g_1(a_2)\Big) + \ldots + \Big(g_n(a_n+h_n) - g_1(a_n)\Big) \\
&= \Big(g_1'(a_1+c_1)\cdot h_1\Big) + \Big(g_2'(a_2+c_2)\cdot h_2\Big) + \ldots + \Big(g_n'(a_n+c_n)\cdot h_n\Big)
\end{aligned}$$

where the third equality follows from the mean value theorem: $f(x) - f(x_0) = f'(\xi)(x - x_0)$ for some $\xi \in (x_0, x)$. Now we know that the values of g_i' are just values of $\partial_i f$, which we know to be bounded (i.e. finite), so $g_1'(a_1+c_1)\cdot h_1 + g_2'(a_2+c_2)\cdot h_2 + \ldots + g_n'(a_n+c_n)\cdot h_n$ just becomes $\mathbf{x}\cdot\mathbf{h}$ for some finite vector $\mathbf{x} \in \mathbb{R}^n$. As $\mathbf{h} \to 0$, $f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) = \mathbf{x}\cdot h \to 0$, so f is continuous at $\mathbf{a}$ for all $\mathbf{a} \in \mathbb{S}$. QED!

Problem §2.3 - 4

We have $u = f(r)$ where $r = |\mathbf{x}| = \sqrt{x_1^2 + \ldots + x_n^2}$. We want to show that $\sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}\right)^2 = [f'(r)]^2$. This is a straightforward application of the chain rule (r is differentiable for $\mathbf{x} \neq \mathbf{0}$, and I guess f is too?)

$$\frac{\partial u}{\partial x_i} = \frac{df}{dr}\frac{\partial r}{\partial x_i} = f'(r)\frac{x_i}{\sqrt{x_1^2 + \ldots + x_n^2}} = f'(r)\frac{x_i}{|\mathbf{x}|}$$

so the sum of these is just

$$\sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}\right)^2 = \frac{[f'(r)]^2}{|\mathbf{x}|^2}\sum_{i=1}^{n} x_i^2 = [f'(r)]^2$$

Problem 7

We have a function $\varphi(x)$ with x's appearing in several places. In the formula for φ, enumerate all the x's with indices $1, \ldots, n$. With this redefinition, we now have a function $F(x_1, x_2, \ldots, x_n)$. In particular, we can recover our original function $\varphi(x)$ as just $\varphi(x) = F(x, x, \ldots, x)$. We see that $\varphi'(x)$ is the directional derivative of F along the vector $(1, 1, \ldots, 1)$. Note that this vector is not a unit vector because slicing F along $(1, 1, \ldots, 1)$ actually results in the function $\varphi(x)$ horizontally stretched out by a factor of $|(1, 1, \ldots, 1)|$, so when we want to find $\varphi'(x)$ we must also scale the unit direction vector by the same factor, but $|(1,1,\ldots,1)| \cdot \frac{(1,1,\ldots,1)}{|(1,1,\ldots,1)|}$ is in fact just $(1, 1, \ldots, 1)$:

$$\varphi'(x) = \nabla f(x, x, \ldots, x)\cdot(1, 1, \ldots, 1) = \frac{\partial F}{\partial x_1}(x, x, \ldots, x) + \ldots + \frac{\partial F}{\partial x_n}(x, x, \ldots, x)$$

or in other words $\varphi'(x)$ is obtained by differentiating with respect to each of the x's (i.e. treating the others as constant) and summing together. This gives rise to the sum and product formulas for derivatives. If we take $\varphi(x) = f(x)^{g(x)}$, $F(x_1, x_2) = f(x_1)^{g(x_2)}$, so

$$\frac{\partial F}{\partial x_1} + \frac{\partial F}{\partial x_2} = f'(x_1) \cdot g(x_2) \cdot f(x_1)^{g(x_2)-1} + g'(x_2) \cdot \ln(f(x_1)) \cdot f(x_1)^{g(x_2)}$$

so

$$\varphi'(x) = f'(x) \cdot g(x) \cdot f(x)^{g(x)-1} + g'(x) \cdot \ln(f(x)) \cdot f(x)^{g(x)}$$

Problem §2.4 - 2

We have a differentiable f on a connected open set S and $\partial_1 f(\mathbf{x}) = 0$ for all $\mathbf{x} \in S$.

(a) If S is convex, then $f(\mathbf{a}) = f(\mathbf{b})$ whenever we have $\mathbf{a}, \mathbf{b} \in S$ where $a_i = b_i$ for all $i \neq 1$: because S is convex and f is differentiable on S, we can apply the mean value theorem to see that $f(\mathbf{b}) - f(\mathbf{a}) = \nabla f(c) \cdot (\mathbf{b} - \mathbf{a})$. But $\mathbf{b} - \mathbf{a}$ is some vector $(d, 0, 0, \ldots, 0)$, and $\nabla f(c)$ has 0 as the first element because $\partial_1 f(\mathbf{x}) = 0$ for all $\mathbf{x} \in S$. This yields $f(\mathbf{b}) - f(\mathbf{a}) = 0$, as desired.

(b) A counterexample for when S is not convex is as follows: take S to be the complement of the closed set $\{(x, y) \in \mathbb{R}_2 : -1 \leq x \leq 1 \text{ and } y \geq 0\}$. For $y < 0$, define f to be 0. For $y \geq 0$ and $x > 1$, take f to be y^2. And for $y \geq 0$ and $x < 1$, take f to be $-y^2$. The picture of this function would look somewhat like a stairwell, starting from the upper left, ascending to a flat landing and continuing on upward by going to the upper right. $f_x(x, y) = 0$ for all $(x, y) \in S$, but $f(x, y) \neq f(-x, y)$.

334 Homework 4

Daniel Rui - 10/17/19

Problem §1.8 - 1

We want to show that Hölder continuity on S (i.e. $|f(x) - f(y)| \leq C|x-y|^\lambda$ for all $x, y \in S$ for positive constants C, λ) implies uniform continuity on S. By the definition of uniform continuity: for all $\epsilon > 0$, we can fix a $\delta > 0$ s.t. for all $x, y \in S$, $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$. Keeping in mind of course that $a < b \implies a^\lambda < b^\lambda$ where $a, b, \lambda > 0$ because x^λ is an increasing function, it is now trivial to see that $|x - y| < \left(\frac{\epsilon}{C}\right)^{1/\lambda}$ in conjunction with Hölder continuity gives

$$|f(x) - f(y)| \leq C|x-y|^\lambda < C\left[\left(\frac{\epsilon}{C}\right)^{1/\lambda}\right]^\lambda = \epsilon$$

Problem 2

More on Hölder continuity:

(a) For $0 < \lambda < 1$, $(a+b)^\lambda < a^\lambda + b^\lambda$, by integration (?!):

$$(a+t)^{\lambda-1} < t^{\lambda-1} \quad (\forall t > 0) \implies \int_0^b (a+t)^{\lambda-1}\,dt < \int_0^b t^{\lambda-1}\,dt \implies (a+b)^\lambda - a^\lambda < b^\lambda$$

(b) Denote $f_\lambda(x) = |x|^\lambda$. We want to prove that $\left||x|^\lambda - |y|^\lambda\right| < |x-y|^\lambda$. By the triangle inequality and part (a), we have that $|x|^\lambda \leq (|x-y| + |y|)^\lambda < |x-y|^\lambda + |y|^\lambda \implies |x|^\lambda - |y|^\lambda < |x-y|^\lambda$. If we focus on y instead, we get $|y|^\lambda - |x|^\lambda < |x-y|^\lambda$ instead. So these two inequalities together yield the desired result. By Problem 1, $f_\lambda(x)$ is uniformly continuous.

Problem 4

We are given that a function $f : S \to \mathbb{R}^m$ is uniformly continuous on S and that $\{x_k\}$ is a Cauchy sequence in S, and we want to prove that that implies $\{f(x_k)\}$ is a Cauchy sequence too. This is a relatively straightforward use of definitions: for all $\epsilon > 0$, we can find one δ s.t. for all $x, y \in S$, $|x-y| < \delta \implies |f(x) - f(y)| < \epsilon$; and for all $n, m \geq N$ for some $N \in \mathbb{N}$, $|x_m - x_n| < \epsilon'$ (for all $\epsilon' > 0$).

Stitching these together, we see that for all ϵ we can find δ s.t. for some N, $n, m \geq N \implies |x_m - x_n| < \delta \implies |f(x_m) - f(x_n)| < \epsilon$ (where again $x_m, x_n \in S$). Hence, $\{f(x_k)\}$ is a Cauchy sequence.

For non-uniformly continuous functions on $(0, \infty)$, consider $f(x) = \frac{1}{k}$ and the Cauchy sequence $x_k = \frac{1}{2^k}$ and correspondingly $f(x_k) = 2^k$. Clearly, $\{f(x_k)\}$ is not a Cauchy sequence.

Problem §2.1 - 1

Suppose that f is differentiable on the interval I and $f'(x) > 0$ for all $x \in I$ except for finitely many points at which $f'(x) = 0$. Time for a small lemma: if $f' > 0$ for all points in a subinterval (a, b), $f(b) - f(a)$ can't possibly be ≤ 0 because if it were, the mean value theorem would imply that for some point in (a, b), f' would also be ≤ 0, which it is not.

Now considering the entire interval I, enumerate the finite set of points where $f' = 0$ as $x_1, \ldots, x_n$. Now for any $x, y \in I$, $x < y$, consider the finite set of subintervals $(x, x_1), (x_1, x_2), \ldots, (x_n, y)$ upon which we know f' is always > 0. From the small lemma above, we know that $f(x_1) - f(x), f(x_2) - f(x_1), \ldots, f(y) - f(x_n)$ are all > 0, and hence the sum of these differences, $f(y) - f(x)$, is also > 0. Because this is true for arbitrary $x, y \in I$, f is strictly increasing.

Problem 2

The function $f(x) = x^2 \sin(\frac{1}{x})$ is differentiable for all $x \in \mathbb{R}$ not equal to 0, and in fact has derivative equal to $f'(x) = 2x \sin(\frac{1}{x}) - \cos(\frac{1}{x})$. At $x = 0$,

$$f'(0) = \lim_{x \to 0} \frac{x^2 \sin(\frac{1}{x})}{x} = 0$$

because $0 \leq \lim_{x \to 0} |x \sin\left(\frac{1}{x}\right)| \leq \lim_{x \to 0} |x| = 0$, and $\lim_{x \to 0} |x \sin\left(\frac{1}{x}\right)| = 0 \implies \lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$. This limit exists and hence f is differentiable everywhere. However, the derivative f' is not continuous at 0 because $\lim\limits_{x \to 0} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$ does not even exist, let alone equal $f'(0) = 0$.

Problem 3

Let us define $g(x) = f(x) + \frac{1}{2}x$. We know that $g'(0) = f'(0) + \frac{1}{2} = \frac{1}{2} > 0$ ($f'(0) = 0$ from above). For $x \neq 0$, $g'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) + \frac{1}{2} \leq 2x$. Conveniently, for every $k \in \mathbb{N}$, $g'(\frac{1}{2\pi k}) = -\frac{1}{2}$, so by the continuity of $g'(x)$ for $x \neq 0$, there must be a open interval (a_k, b_k) around each of the $x = \frac{1}{2\pi k}$ for which $g' < 0$. For any $a, b \in (a_k, b_k), a < b$, $g(b) > g(a)$ (proof by contradiction using MVT), so for arbitrarily small $\epsilon > 0$, I can find some $x = \frac{1}{2\pi k}$ for which g is decreasing on $(a_k, b_k) \subset (x - \epsilon, x + \epsilon)$, as desired.

Problem 4

Now we consider the function $h(x)$ which is x^2 when $x \in \mathbb{Q}$ and 0 otherwise. Clearly, h is discontinuous for all $x \neq 0$ because then we can choose $\epsilon = \frac{x}{2}$ and for all neighborhoods around x there will always be a rational/irrational x' "close enough" s.t. $|h(x) - h(x')| > \frac{x}{2}$ (by denseness of rationals and irrationals) — thus h is most definitely not differentiable for $x \neq 0$. On the other hand at $x = 0$, the limit $\lim\limits_{x \to 0} \frac{h(x)}{x} \leq \lim\limits_{x \to 0} \frac{x^2}{x} = 0$ exists (and equals 0) and hence h is in fact only differentiable at $x = 0$.

Problem 8

We suppose that $\mathbf{f}, \mathbf{g} : \mathbb{R} \to \mathbb{R}^n$. We can represent $\mathbf{f}$ for example as $(f_1, \ldots, f_n)$ for real-valued functions f_i. Then $(\mathbf{f} \cdot \mathbf{g})' = (f_1 g_1 + \ldots + f_n g_n)' = (f_1' g_1 + f_1 g_1') + \ldots (f_n' g_n + f_n g_n') = \mathbf{f}' \cdot \mathbf{g} + \mathbf{f} \cdot \mathbf{g}'$.

Similarly, if $n = 3$,

$$\begin{aligned}
(\mathbf{f} \times \mathbf{g})' &= \langle (f_2 g_3 - f_3 g_2)', -(f_1 g_3 - f_3 g_1)', (f_1 g_2 - f_2 g_1)' \rangle \\
&= \langle (f_2' g_3 + f_2 g_3' - f_3' g_2 - f_3 g_2'), (-f_1' g_3 - f_1 g_3' + f_3' g_1 + f_3 g_1'), (f_1' g_2 + f_1 g_2' - f_2' g_1 - f_2 g_1') \rangle \\
&= \langle (f_2' g_3 - f_3' g_2), (-f_1' g_3 + f_3' g_1), (f_1' g_2 - f_2' g_1) \rangle + \langle (f_2 g_3' - f_3 g_2'), (-f_1 g_3' + f_3 g_1'), (f_1 g_2' - f_2 g_1') \rangle \\
&= \mathbf{f}' \times \mathbf{g} + \mathbf{f} \times \mathbf{g}'
\end{aligned}$$

Truly horrible.

Problem 9

We are given the function $f(x)$ which equals e^{-1/x^2} for $x \neq 0$ and 0 at $x = 0$.

(a) We want to find $\lim\limits_{x \to 0} \frac{f(x)}{x^n}$ for any $n > 0$. Make the substitution $y = 1/x^2$; the limit now becomes $\lim\limits_{y \to \infty} y^{n/2} e^{-y}$ (sketch of proof: $|x| < \delta \implies |1/x^2| > N \implies |y| > N \implies |y^{n/2} e^{-y}| < \epsilon$). We know from Corollary 2.12 in the book that this limit is equal to 0.

(b) To show that f is differentiable at $x = 0$, we just need to show the limit $\lim\limits_{x \to 0} \frac{f(x)}{x}$ exists, which is just part (a) at $n = 1$. Furthermore, part (a) also gives that $f'(0) = 0$.

(c) For $x \neq 0$, $f'(x) = 2x^{-3} e^{-1/x^2}$, which is indeed of the form $P(1/x) e^{-1/x^2}$ where $P(1/x)$ is a polynomial of degree 3. Now assume that $f^{(k)} = P_k(1/x) e^{-1/x^2}$ where $P_k(1/x)$ is a polynomial of degree $3k$. Then, $f^{(k+1)} = [P_k(1/x)(2x^{-3}) + (-x^{-2}) Q_k(1/x)] e^{-1/x^2}$ where Q is the derivative of P and is therefore a polynomial in $1/x$ with degree $3k+1$. We see that $P_{k+1} = P_k(1/x)(2x^{-3}) + (-x^{-2}) Q_k(1/x)$ is also a polynomial in $1/x$ with degree $3k+3$, which completes the induction.

(d) $f^{(k+1)}(0) = \lim\limits_{x \to 0} \dfrac{f^{(k)}(x)}{x} = P_k'(1/x) e^{-1/x^2}$, where $P_k'(1/x) = \frac{1}{x} P(1/x)$ is just another polynomial in $1/x$. We can now split this polynomial into its individual terms (times of course the e^{-1/x^2}) and take the limits of those. From part (a), we know that the limits of the individual terms are all 0, so in total the limit is 0, so $f^{(k+1)}(0) = 0$ for all $k \in \mathbb{N}$.

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Problem §1.5 - 3

If $a = 0$, the sequence $x_k = \frac{1}{k}$ will converge to 0. If $a = 1$, the sequence $x_k = 1 - \frac{1}{k} = \frac{k-1}{k}$ will converge to 1. If $a \in (0, 1)$, let x_k be $\dfrac{\lfloor 10^k \cdot a \rfloor}{10^k}$ (i.e. the truncation of the first k digits right of the decimal point divided by 10^k).

Problem 5

We have a sequence defined by $x_1 = \sqrt{2}$, $x_{k+1} = \sqrt{2 + x_k}$. We prove that it's monotonically increasing and bounded and hence convergent.

(a) As a base case, $x_1 = \sqrt{2} < 2$ (I'm actually going to go further and say that $0 < x_1 < 2$). Now assume $0 < x_k < 2$; then

$$0 < x_k + 2 < 4 \implies 0 < \sqrt{x_k + 2} < 2 \implies 0 < x_{k+1} < 2$$

(b) Because $x_{k+1} = \sqrt{2 + x_k}$ for all $k \geq 1$, that means that $x_{k+1}^2 = x_k + 2 > x_k + x_k = 2x_k > x_k^2$ which means that $x_{k+1} > x_k$ for all $k \geq 1$.

(c) We know now that x_k is monotone increasing and bounded, so by the monotone convergence theorem (MCT), x_k converges to some limit L.

I'm now going to prove a small lemma that will help in the future when finding limits explicitly: [if a sequence is defined as $x_{k+1} = f(x_k)$ where $x_k \to L$ and f is continuous at L, then $L = f(L)$].

The proof goes like this: f is continuous at L means that $\forall \epsilon, \exists \delta$ such that $|x - L| < \delta \implies |f(x) - f(L)| < \epsilon$, but $x_k \to L$ means that $\forall \epsilon', \exists K$ such that $k \geq K \implies |x_k - L| < \epsilon'$. Take $\epsilon' = \delta$ (the δ from the continuity of f); then we have that $k \geq K \implies |f(x_k) - f(L)| < \epsilon$. Because $x_{k+1} - f(x_k) = 0$, we can write

$$|L - f(L)| = |L - f(L) - (x_{k+1} - f(x_k))| = |(L - x_{k+1}) + (f(x_k) - f(L))|$$

which by the triangle inequality, is less than or equal to $|L - x_{k+1}| + |f(x_k) - f(L)| < \epsilon' + \epsilon$ (for sufficiently large k). But because the epsilons were arbitrarily small, $|L - f(L)|$ is arbitrarily small, so $L = f(L)$.

Using this lemma, we see that the x_k do in fact have a limit L, and $f(x) = \sqrt{2 + x}$ is continuous (by limit properties discussed in previous homeworks/sections of the book), we see that $L = \sqrt{2 + L} \implies L = 2, -1$. But $L \neq -1$ because L must be the supremum (by the MCT), so $L = 2$.

Problem 6

Given: $F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$ and $r_k = \frac{F_{k+1}}{F_k}$

(a)

$$\frac{r_k + 1}{r_k} = \frac{\frac{F_{k+1}}{F_k} + 1}{\frac{F_{k+1}}{F_k}} = \frac{\frac{F_{k+1} + F_k}{F_k}}{\frac{F_{k+1}}{F_k}} = \frac{F_{k+1} + F_k}{F_{k+1}} = \frac{F_{k+2}}{F_{k+1}} = r_{k+1}$$

$$\frac{2r_k + 1}{r_k + 1} = \frac{\frac{2F_{k+1}}{F_k} + 1}{\frac{F_{k+1}}{F_k} + 1} = \frac{\frac{2F_{k+1} + F_k}{F_k}}{\frac{F_{k+1} + F_k}{F_k}} = \frac{F_{k+1} + F_{k+1} + F_k}{F_{k+1} + F_k} = \frac{F_{k+1} + F_{k+2}}{F_{k+2}} = \frac{F_{k+3}}{F_{k+2}} = r_{k+2}$$

(b) We want to prove that $r_{2k-1} < r_{2k+1} < \varphi < r_{2k+2} < r_{2k}$ for all $k \geq 1$. As a base case, we know that $r_1 < r_3 < \varphi < r_4 < r_2$ by just verifying numerically that $1 < \frac{3}{2} < \varphi < \frac{5}{3} < 2$. Now assuming that $r_{2k-1} < r_{2k+1} < \varphi < r_{2k+2} < r_{2k}$ is true, let's prove it's true for $(k+1)$: $r_{2k+1} < r_{2k+3} < \varphi < r_{2k+4} < r_{2k+2}$.

From our assumption, we already have the outer bounds: $r_{2k+1} < \varphi < r_{2k+2}$, so we just want to prove that $r_{2k+1} < r_{2k+3} < \varphi$ and $\varphi < r_{2k+4} < r_{2k+2}$.

We can check comparisons with φ by seeing that $x < \varphi \iff x^2 < x + 1$ and $x > \varphi \iff x^2 > x + 1$, with help from the second identity from part (a). Logic going right to left:

$$\begin{aligned}
r_{2k+3} < \varphi \iff r_{2k+3}^2 < r_{2k+3} + 1 &\iff \left(\frac{2r_{2k+1} + 1}{r_{2k+1} + 1}\right)^2 < \frac{2r_{2k+1} + 1}{r_{2k+1} + 1} + 1 \\
&\iff (2r_{2k+1} + 1)^2 < (2r_{2k+1} + 1)(r_{2k+1} + 1) + (r_{2k+1} + 1)^2 \\
&\iff 4r_{2k+1}^2 + 4r_{2k+1} + 1 < 2r_{2k+1}^2 + 3r_{2k+1} + 1 + r_{2k+1}^2 + 2r_{2k+1} + 1 \\
&\iff r_{2k+1}^2 < r_{2k+1} + 1 \iff r_{2k+1} < \varphi
\end{aligned}$$

and

$$\begin{aligned}
r_{2k+4} > \varphi \iff r_{2k+4}^2 > r_{2k+4} + 1 &\iff \left(\frac{2r_{2k+2} + 1}{r_{2k+2} + 1}\right)^2 > \frac{2r_{2k+2} + 1}{r_{2k+2} + 1} + 1 \\
&\iff (2r_{2k+2} + 1)^2 > (2r_{2k+2} + 1)(r_{2k+2} + 1) + (r_{2k+2} + 1)^2 \\
&\iff 4r_{2k+2}^2 + 4r_{2k+2} + 1 > 2r_{2k+2}^2 + 3r_{2k+2} + 1 + r_{2k+2}^2 + 2r_{2k+2} + 1 \\
&\iff r_{2k+2}^2 > r_{2k+2} + 1 \iff r_{2k+2} > \varphi
\end{aligned}$$

The final pieces:

$$\begin{aligned}
r_{2k+1} < r_{2k+3} &\iff r_{2k+1} < \frac{2r_{2k+1} + 1}{r_{2k+1} + 1} \iff r_{2k+1}^2 + r_{2k+1} < 2r_{2k+1} + 1 \\
&\iff r_{2k+1}^2 < r_{2k+1} + 1 \iff r_{2k+1} < \varphi
\end{aligned}$$

and

$$r_{2k+2} > r_{2k+4} \iff r_{2k+2} > \frac{2r_{2k+2}+1}{r_{2k+2}+1} \iff r_{2k+2}^2 + r_{2k+2} > 2r_{2k+2}+1$$
$$\iff r_{2k+2}^2 > r_{2k+2}+1 \iff r_{2k+2} > \varphi$$

(c) From the above, r_k for odd k is a monotonically increasing bounded sequence, so it has a limit L that obeys $L = \frac{L+1}{L} \iff L^2 = L+1 \iff L = \varphi$ (because of the first identity of part (a) and my lemma from Problem 5 of this section). Similarly, the above tells us that r_k for even k is a monotonically decreasing bounded sequence, so it has a limit L that obeys $L = \frac{L+1}{L} \iff L^2 = L+1 \iff L = \varphi$. QED!

Problem 9

We define the lower and upper limits to be $\liminf_{n\to\infty} x_n := \lim_{n\to\infty}\left(\inf_{k\geq n} x_k\right)$, and $\limsup_{n\to\infty} x_n := \lim_{n\to\infty}\left(\sup_{k\geq n} x_k\right)$. In this problem I will omit the $n \to \infty$ when referring to $\liminf_{n\to\infty} x_n$ or $\limsup_{n\to\infty} x_n$.

By the book (proof using the monotone convergence theorem), both these limits exist. Now we just need to prove that $\limsup x_k = a \iff$ there are infinitely many $x_k > a - \epsilon$ but only finitely many $x_k > a + \epsilon$ for every $\epsilon > 0$.

$(\implies)$ Assume that $\limsup x_k = a$ but there are only finitely many $x_k > a - \epsilon$. Then past a certain point, all the $x_k \leq a - \epsilon$, meaning the supremum for those x_k past that point could not possibly be greater than $a - \epsilon$; but $\limsup x_k = a > a - \epsilon$ so contradiction. If there are infinitely many $x_k > a + \epsilon$ then no matter how big I go there will always be an $x_k > a + \epsilon$, so it would be impossible for any of the $\left(\sup_{k\geq n} x_k\right)$ to be less than $a + \epsilon$; but $\limsup x_k = a < a + \epsilon$ so contradiction

$(\impliedby)$ If there are only finitely many $x_k > a + \epsilon$ for every $\epsilon > 0$, that means that after a certain point all of the $x_k \leq a + \epsilon$; hence, $\left(\sup_{k\geq n} x_k\right)$ for x_k past that point can not be greater than $a + \epsilon$. If there are infinitely many $x_k > a - \epsilon$ for ever $\epsilon > 0$, that means that no matter how big I go I will always be able to find $x_k > a - \epsilon$, so the supremum can never be less than $a - \epsilon$. Hence, we have that $a - \epsilon \leq \limsup x_k \leq a + \epsilon$ for all ϵ, and hence $\limsup x_k = a$.

The analogous condition for $\liminf x_k = a$ would be that there are infinitely many $x_k < a + \epsilon$ and finitely many $x_k < a - \epsilon$ for every ϵ.

Problem §1.6 - 1

From the section on continuity, if $f : D \to R$ is continuous and and a set U is open/closed, then $S = \{x \in D : f(x) \in U\}$ is open/closed respectively (Theorem 1.13). That means for our problem, a very safe bet would be to choose $\mathbb{R}$ to be the domain, because $\mathbb{R}$ is both open and closed.

(a) The continuous function e^x takes the closed set $\mathbb{R}$ to the open interval (and thus open set) $(0, \infty)$. (The book stated the continuity of e^x (albeit without proof), so I will accept it as a given).

(b) $[0, \infty)$ is a closed, because its complement $(-\infty, 0)$ is open. The continuous function $|x|$ takes the closed set $\mathbb{R}$ to the aforementioned $[0, \infty)$.

Problem 2

Continuous functions on bounded sets:

(a) An obvious example of an unbounded function (continuous on $\mathbb{R} \setminus \{0\}$) on a bounded interval is that of $f(x) = \frac{1}{x}$ on $(0, 1)$ — it's continuous on the entire interval (by limit/continuity properties because x never equals 0), but for every $N \in \mathbb{N}$, $f(\frac{1}{2N}) > N$, so f is unbounded.

(b) However, if $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous everywhere, then $S \subset \mathbb{R}^n$ is bounded implies $f(S)$ is bounded. Before we proceed, let me prove (as a mini-lemma?) that if two open balls satisfy $B(r_1, x_0) \subset B(r_2, x_0)$, then $B(r_1, x_0) \subset K \subset B(r_2, x_0)$ where K is a closed set defined as $\{x \in \mathbb{R}^n : |x - x_0| \leq r_1\}$. The proof is just two steps: one, prove that K is closed (by proving the complement is open: for any $x \in K^C$, the open ball with radius $\frac{|x-x_0|-1}{2}$ is disjoint from K); and two, $|x - x_0| < r_1 \implies |x - x_0| \leq r_1 \implies |x - x_0| < r_2 \implies B(r_1, x_0) \subset K \subset B(r_2, x_0)$.

The proof of the problem goes as follows: if S is bounded, then there is some open ball B_1 with finite radius s.t. $S \subset B_1$. Then find another ball B_2 with finite radius greater than that of B_1. Then by my mini-lemma, $S \subset B_1 \subset K \subset B_2$. K is closed and bounded (bounded by open ball B_2), so K is a compact set that encloses S.

We know that continuous functions take compact sets to compact sets (Theorem 1.22 in the book, proven using Bolzano-Weierstrass and sequences), so $f(K)$ is compact as well. But clearly $x \in S \subset K \implies f(x) \in f(S)$ and $x \in f(K)$ for all $x \in S$, which means that $f(S) \subseteq f(K)$, so $f(S)$ must be bounded because $f(K)$ is compact and hence bounded.

Problem 4

If $S \in \mathbb{R}^n$ is compact and $f : S \to \mathbb{R}$ is continuous and $f(x) > 0$ for all $x \in S$, then $f(x) \geq c$ for all $x \in S$ for some fixed $c > 0$. We prove this by contradiction: suppose there is no such c; then for every $k \in \mathbb{N}$ we will be able to find some x_k satisfying $f(x_k) < \dfrac{1}{k}$.

By Bolzano-Weierstrass, we are guaranteed a subsequence x_j (where $j \in$ some subset of $\mathbb{N}$) that converges to a point $x \in S$, or in math notation: for every $\delta > 0$, there is some $J \in \mathbb{N}$ s.t. $j \geq J \implies |x_j - x| < \delta$. From the continuity of f, we know that for every $\epsilon' > 0$ we can find δ (i.e. find a J) such that $j \geq J \implies |x_j - x| < \delta \implies |f(x) - f(x_j)| < \epsilon'$.

The triangle inequality gives us that $|f(x)| \leq |f(x) - f(x_j)| + |f(x_j)| < \epsilon' + \frac{1}{j}$, but because this is true for arbitrarily small ϵ' and $\frac{1}{j}$, we see that $|f(x)| = f(x)$ is arbitrarily small, implying that it is 0. We just found a point $x \in S$ s.t. $f(x) = 0$, so we have reached a contradiction; there must exist a lower bound c.

Problem 6

Define the notion of distance between two sets A, B to be $d(A,B) = \inf\{|\mathbf{a}-\mathbf{b}| : \mathbf{a} \in A, \mathbf{b} \in B\}$

(a) Prove that if one of U, V contain a point in the other's closure, then $d(U,V) = 0$. w.l.o.g., assume U has a point x in the closure of V. Because x is in $\bar{V} = V^{\text{int}} \cup \partial V$, we know that every ϵ-ball around x has a non-empty intersection with V, i.e. for every ϵ, there is a point $v \in V$ s.t. $|x-v| < \epsilon$, which means that $d(U,V) = \inf\{|\mathbf{a}-\mathbf{b}| : \mathbf{a} \in U, \mathbf{b} \in V\} \leq |x-v| < \epsilon \implies d(U,V) = 0$.

(b) Prove that if U is compact, V is closed, $U \cap V = \varnothing$, then $d(U,V) > 0$. We do this by contradiction: assume that U is compact (closed and bounded), V is closed, $U \cap V = \varnothing$, and $d(U,V) = 0$. The distance being 0 implies for every $k \in \mathbb{N}$ I can find $\mathbf{u}_k \in U, \mathbf{v}_k \in V$ s.t. $|\mathbf{u}_k - \mathbf{v}_k| < \frac{1}{k}$.

By Bolzano-Weierstrass, every sequence of a compact set (U in our case) has a subsequence that converges to a point in the set, which means that subsequence $\mathbf{u}_j$ ($j \in$ some subset of $\mathbb{N}$) converges to $\mathbf{u}$ for some point $\mathbf{u} \in U$, implying that for all $\epsilon' > 0$, there is some $J \in \mathbb{N}$ s.t. for all $j \geq J$, $|\mathbf{u} - \mathbf{u}_j| < \epsilon'$. By the triangle inequality, $|\mathbf{u} - \mathbf{v}_j| \leq |\mathbf{u} - \mathbf{u}_j| + |\mathbf{u}_j - \mathbf{v}_j| < \frac{1}{j} + \epsilon'$. This is true for all ϵ' and $\frac{1}{j}$, so we will always be able to say $|\mathbf{u} - \mathbf{v}_j| < \epsilon$ for any $\epsilon > 0$, which means that $\mathbf{v}_j$ converge to $\mathbf{u}$.

From Problems §1.4 − 6, 7 from the previous homework, a sequence of points in S of tending towards to a limit L implies that $L \in \bar{S}$. If S is closed, then $\bar{S} = S \cup \partial S \subseteq S$ (because closed sets contain all the boundary points), so $L \in S$. We can apply this lemma to our current situation, where $\mathbf{v}_j \to \mathbf{u} \implies \mathbf{u} \in V$. If $\mathbf{u} \in U$ is *also* in V, then $U \cap V \neq \varnothing$; contradiction. Therefore, the distance can't be 0, so $d(U,V) > 0$.

(c) The above doesn't hold true if both sets are just closed (not necessarily bounded). Consider the closed sets $A = \{(x,y) : y \geq e^{-x}\}$ and $B = \{(x,y) : y \leq -e^{-x}\}$ (it's a straightforward proof to prove the complements of these sets are open by using continuity to find an $\epsilon - \delta$ pair for every point $\implies$ the open ball of radius $\min\{\epsilon, \delta\}$ is disjoint from the set — I did this in Homework 2 (§1.2 - 1e) so I will not go into any further details).

It's pretty clear that $A \cap B = \varnothing$ because there is no (x,y) s.t. y is both $y \geq e^{-x} > 0$ and $y \leq -e^{-x} < 0$, but for any $\epsilon > 0$, there is some N s.t. $e^{-N} < \frac{\epsilon}{2}$, so $d(A,B) = \inf\{|\mathbf{a}-\mathbf{b}| : \mathbf{a} \in A, \mathbf{b} \in B\} \leq |(N, e^{-N}) - (N, -e^{-N})| = 2e^{-N} < \epsilon$. This is true for any ϵ, so $d(A,B) = 0$.

Problem §1.7 - 2

Suppose we have two points $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ contained within the set $S = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = 1\}$ (i.e. on the surface of a n-dimensional sphere). Also note that we are using the Euclidean norm. Now take the function $f : [0,1] \to \mathbb{R}^n = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$ which, from $t = 0$ to $t = 1$, traces a (continuous) straight line from point $\mathbf{a}$ to point $\mathbf{b}$, passing through the sphere. As is stands, this line does not lie in S; however, we can easily make it so that it does, by normalizing every point on the line:

$$f(t) = \frac{\mathbf{a} + t(\mathbf{b}-\mathbf{a})}{|\mathbf{a} + t(\mathbf{b}-\mathbf{a})|} = \frac{\mathbf{a} + t(\mathbf{b}-\mathbf{a})}{\sqrt{\big[(1-t)\mathbf{a} + t\mathbf{b}\big] \cdot \big[(1-t)\mathbf{a} + t\mathbf{b}\big]}} = \frac{\mathbf{a} + t(\mathbf{b}-\mathbf{a})}{\sqrt{(1-t)^2 \mathbf{a}\cdot\mathbf{a} + 2t(1-t)\mathbf{a}\cdot\mathbf{b} + t^2\mathbf{b}\cdot\mathbf{b}}}$$

but because $\mathbf{a}$ and $\mathbf{b}$ are on the sphere, $\mathbf{a}\cdot\mathbf{a} = \mathbf{b}\cdot\mathbf{b} = 1$, so

$$f(t) = \frac{\mathbf{a} + t(\mathbf{b}-\mathbf{a})}{\sqrt{\big(1-2t+2t^2\big) + \big(2t-2t^2\big)\mathbf{a}\cdot\mathbf{b}}}$$

By limit/continuity properties, f is continuous if the denominator is not 0; hence, we must find when the denominator is 0 (for $t \in [0,1]$). Denoting $c = \mathbf{a}\cdot\mathbf{b}$ for ease (keeping in mind $|c| \le 1$, easily provable by Cauchy-Schwarz), then $1 - 2t + 2t^2 + 2ct - 2ct^2 = 0 \iff (2c-2)t^2 - (2c-2)t - 1 = 0$, which we then can use the quadratic formula on, yielding

$$\begin{aligned} t &= \frac{2c-2 \pm \sqrt{(2c-2)^2 + 4(2c-2)}}{2(2c-2)} = \frac{c-1 \pm \sqrt{(c-1)^2 + 2c-2}}{2c-2} = \frac{1}{2} \pm \frac{\sqrt{c^2-1}}{2c-2} \\ &= \frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{c+1}{c-1}} = \frac{1}{2} \pm \frac{1}{2}\sqrt{-1 + \frac{2c}{c-1}} \end{aligned}$$

Let us ignore when $c = \mathbf{a}\cdot\mathbf{b} = 1$ (i.e. when $\mathbf{a} = \mathbf{b}$, which we don't care about because we don't care about "paths" from a point to itself). Given this, we see now that $c < 1 \iff c - 1 < 0$ (so we flip the inequalities if we multiply by it). Finally, $t = \frac{1}{2} \pm \frac{1}{2}\sqrt{-1 + \frac{2c}{c-1}}$ which attains real values when $\frac{2c}{c-1} \ge 1 \iff 2c \le c-1 \iff c \le 1$. But because $|c| \le 1$, the only c we need to pay attention to is $c = -1$. Plugging c in to our equation for t, we see that f is discontinuous at $t = 1/2$ when $c = -1$, or when $\mathbf{a}$ and $\mathbf{b}$ are opposite each other.
Because f is continuous for all other values of c (all other orientations of $\mathbf{a}$ and $\mathbf{b}$), we can solve this edge case easily by finding a connected path to from $\mathbf{a}$ to some $\mathbf{c} \ne \mathbf{b}$ and then a path $\mathbf{c}$ to $\mathbf{b}$ and connecting the two paths (parameterizing t in the first function to range from $[0, \frac{1}{2}]$ and t in the second function to range from $[\frac{1}{2}, 1]$ so the total function has t from $[0,1]$).

[Interjection:] Or really, it would have been easier to prove that all norms are continuous functions: $|\mathbf{a}-\mathbf{b}| < \epsilon \implies \big||\mathbf{a}| - |\mathbf{b}|\big| < \epsilon$ (literally just the reverse triangle inequality). In the above proof, that would imply the continuity of f (except when the norm is 0) for *any general* norm, without the need for a single square root anywhere.

Problem 3

Given a continuous, one-to-one function $f : I \to \mathbb{R}$ where I is some interval, we want to prove that it's strictly increasing or strictly decreasing. Suppose that there is a continuous, one-to-one function on an interval I that is not strictly increasing nor decreasing; then for $x < y < z$, we will have either $f(x) \le f(z) \ge f(y)$ or $f(x) \ge f(z) \le f(y)$. w.l.o.g., we will look at the first case. If any two of $f(x), f(z), f(y)$ equal each other, then that violates one-to-one, so we now only consider the cases where $f(x) < f(z) \ge f(y)$. w.l.o.g., let's say that $f(x) > f(y)$. So if I find a t s.t. $f(x) < t < f(z)$, then it will also be the case that $f(y) < t < f(z)$. By the IVT (Corollary 1.27), there is a $c_1 \in (x, y)$ and $c_2 \in (y, z)$ s.t. $f(c_1) = t = f(c_2)$, which again violates one-to-one. Hence, the function must be strictly increasing or decreasing.

Problem 4

A set S is disconnected if I can find non-empty A, B such that $A \cup B = S$, $A \cap \bar{B} = \varnothing$, and $\bar{A} \cap B = \varnothing$. We want to prove that if S_1, S_2 are connected (and have at least one point in common), then $S_1 \cup S_2$ is connected as well. We do this by contradiction: assume that $S_1 \cup S_2$ is disconnected, i.e. that we can find A, B s.t. the criterion hold. Then, we can decompose S_1 into $(S_1 \cap A) \cup (S_1 \cap B)$ (the $\supseteq$ direction is obvious, and the $\subseteq$ direction is true because $S_1 \subset S = A \cup B$ so every point of S_1 must be in either A or B).

Mini-lemma: if a set $U \subseteq V$, then $\bar{U} \subseteq \bar{V}$ — using Problems §1.4 - 6,7 from Homework 2, every boundary point of U is an accumulation point, and every accumulation point of U has a sequence in U that converges to that point, but every sequence in U is also a sequence in V, so that sequence must also converge to an accumulation point in V, and hence be in $\bar{V}$.

With this mini-lemma, we see that $(S_1 \cap A) \subseteq A \implies \overline{(S_1 \cap A)} \cup (S_1 \cap B) \subseteq \bar{A} \cup (S_1 \cap B) \subseteq \bar{A} \cup B = \varnothing$, so $\overline{(S_1 \cap A)} \cup (S_1 \cap B) = \varnothing$ (the other case is analogous), so in fact if $(S_1 \cap A)$ and $(S_1 \cap B)$ are non-empty, then this is a disconnection of a connected set S_1, which is impossible. Thus, the only cases remaining are when one of those two sets is empty; however, w.l.o.g., $S_1 \cap A$ being empty would imply that $A \subset S_2$ (otherwise, S_2 would have to be $\subset A$, but S_1 and S_2 have at least one point in common, so that would imply that the intersection be non-empty). But a similar analysis over on $S_2 = (S_2 \cap A) \cup (S_2 \cap B)$ would imply that $A \subset S_1$! But S_1 is disjoint from S_2 ($S_1 \cap S_2 \subseteq \bar{S}_1 \cap S_2 = \varnothing$), so our remaining cases also end in contradiction, forcing $S_1 \cup S_2$ to be connected. QED

The intersection need not be connected — consider the (pathwise) connected curves $y = x$ and $y = x^2$; the intersection of these two sets is $(0, 0) \cup (1, 1)$ which is very obviously not connected.

Problem 8

Prove that the closure of a connected set S is connected. Very similar to the above problem, we assume that $\bar{S}$ is disconnected and can be split by sets A, B. Then as above, we find that $(S \cap A) \cup (S \cap B)$ is a disconnection of S, a contradiction unless one of them is empty. But w.l.o.g. if $S \cap A = \varnothing$, so S must be $\subseteq B$ (otherwise, there would exist points in S outside of B, and not in A either, contradicting $S \subseteq \bar{S} = A \cup B$). Using my mini-lemma from the above problem, we see that $S \subseteq B \implies \bar{S} \subseteq \bar{B}$, and so we can say that $A = A \cap \bar{S} \subseteq A \cap \bar{B} = \varnothing$, a contradiction. Hence, $\bar{S}$ must be connected.

Problem 9

Given a continuous function f from the surface of a sphere S to $\mathbb{R}$, consider the function $g(\mathbf{x}) = f(\mathbf{x}) - f(-\mathbf{x})$. Picking a point on the sphere, $\mathbf{s}$, $g(\mathbf{s})$ is either $= 0, > 0$, or < 0. If $g(\mathbf{s}) = 0$, then f assumes the same value at two diametrically opposite points, $\mathbf{s}$ and $-\mathbf{s}$. Else, w.l.o.g. assume $g(\mathbf{s}) = f(\mathbf{s}) - f(-\mathbf{s}) > 0$. Then $g(-\mathbf{s}) = -(f(\mathbf{x}) - f(-\mathbf{x})) < 0$. By the IVT (Corollary 1.27 in the text) and by the fact that S is connected (see Problem 2 of this very section), g must attain 0 at some point $\mathbf{s}' \in S$. Then, f assumes the same value at two diametrically opposite points, $\mathbf{s}'$ and $-\mathbf{s}'$.

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Daniel Rui - 10/3/19

Problem §1.2 - 1e

$$S = \left\{(x,y) : x > 0 \text{ and } y = \sin\left(\frac{1}{x}\right)\right\}$$

(i) I'll draw it on the right margin.

(ii) S is neither open nor closed. Using the definition in the book that open $\iff$ S contains NONE of the boundary points and that closed $\iff$ S contains ALL of the boundary points, all we have to do is find one boundary point in S and one not in S.

- $(0,0) \in \partial S$ because all ϵ-balls around $(0,0)$ must have points from S because I can always find $x_0 = \frac{1}{2\pi n} < \frac{\epsilon}{2}$ for some $n \in \mathbb{N}$. $\sin\left(\frac{1}{x_0}\right) = 0$, so the point $(\frac{1}{2\pi n}, 0)$ is in the ϵ-ball. Clearly $(0,0) \in S^C$ so every ϵ-ball around $(0,0)$ contains points in S and S^C, so $(0,0)$ is in the boundary.
- $(\frac{2}{\pi}, 1) \in \partial S$ because $(\frac{2}{\pi}, 1) \in S$ but $(\frac{2}{\pi}, 1 + \frac{\epsilon}{2}) \notin S$ because $\sin(x)$ does not exceed 1, so every ϵ-ball around $(\frac{2}{\pi}, 1)$ contains points in S and S^C.

(iii) Split up into three parts: $S^{\text{int}} = \varnothing$, $\partial S = S \cup \{(0,y) : y \in [-1,1]\}$, $\bar{S} = S \cup \{(0,y) : y \in [-1,1]\}$

(a) For every $(x,y) \in S$, $(x, y - \epsilon) \notin S$ because $\sin(x)$ is a function and can't take two y-values for the same x. So every ball around ever point in S has points in S and S^C, so $S^{\text{int}} = \varnothing$

(b) Every ball around every $(x,y) \in S$ contains that point (x,y) and $(x, y + \frac{\epsilon}{2})$ so all of S is contained in ∂S. Furthermore, I can always find an $x_0 = \dfrac{1}{2\pi n + \arcsin(y_0)} < \frac{\epsilon}{2}$ for some large enough $n \in \mathbb{N}$, so every ball around every point $(0, y_0)$ for $y_0 \in [-1,1]$ contains the point $(0, y_0) \notin S$ and the point $(x_0, y_0) \in S$, and hence every point $(0, y_0)$ for $y_0 \in [-1,1]$ is also in ∂S.

So far we've proved that $S \cup \{(0,y) : y \in [-1,1]\} \subseteq \partial S$. Now we just need to prove that no other points are on the boundary.

- If $x_0 < 0$ for some point (x_0, y_0), then the ball with $r = \dfrac{|x_0|}{2}$ is disjoint from S.
- If $x_0 = 0, |y_0| > 1$, then the ball with $r = \dfrac{|y_0| - 1}{2}$ is disjoint from S.
- If $x_0 > 0$, we know that $f(x) = \sin\left(\frac{1}{x}\right)$ is continuous, so for every ϵ, we can find δ such that $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$. If we choose $\epsilon = \dfrac{|f(x_0) - y_0|}{2}$, then we are guaranteed that for any $x \in (x_0 - \delta, x_0 + \delta)$, all the corresponding y-values will be at least a distance of ϵ away. So the ball $B(\min\{\delta, \epsilon\}, (x_0, y_0))$ will be disjoint from the curve, and hence there are no other boundary points.

(c) $\bar{S}$ is defined to be $S \cup \partial S$, so $\bar{S} = S \cup \{(0,y) : y \in [-1,1]\}$

Problem 1g

$$S = \left\{(x,y) : x, y \in \mathbb{Q}^{[0,1]\times[0,1]}\right\}$$

(i) The picture would look like a black unit square because the rationals are dense:

(ii) S is neither open nor closed. As above, all we have to do is find one boundary point in S and one not in S.

- $(a,b) \in \partial S$ (for irrational a and b in $[0,1]$) because every ϵ-ball around that point must contain that point, which is not in S, and also some point (the "nearest" multiple of $\frac{1}{n}$ to the point) $\left(\frac{\lfloor na \rfloor}{n}, \frac{\lfloor nb \rfloor}{n}\right) \in S$ such that

$$\sqrt{\left(a - \frac{\lfloor na \rfloor}{n}\right)^2 + \left(b - \frac{\lfloor nb \rfloor}{n}\right)^2} = \frac{1}{n}\sqrt{(na - \lfloor na \rfloor)^2 + (nb - \lfloor nb \rfloor)^2} \leq \frac{1}{n}\sqrt{1^2 + 1^2} < \frac{2}{n} < \epsilon$$

  which is satisfied by choosing $n \in \mathbb{N}$ where $n > \frac{2}{\epsilon}$

- Now any $(a,b) \in S$ is also in ∂S because every ϵ-ball around that point must contain that point, which is in S, and also some point $\left(\frac{\lfloor \frac{n}{\sqrt{2}} a \rfloor}{\frac{n}{\sqrt{2}}}, \frac{\lfloor \frac{n}{\sqrt{2}} b \rfloor}{\frac{n}{\sqrt{2}}}\right) \notin S$ such that

$$\sqrt{\left(a - \frac{\lfloor \frac{n}{\sqrt{2}} a \rfloor}{\frac{n}{\sqrt{2}}}\right)^2 + \left(b - \frac{\lfloor \frac{n}{\sqrt{2}} a \rfloor}{\frac{n}{\sqrt{2}}}\right)^2} = \frac{1}{n}\sqrt{\left(\frac{n}{\sqrt{2}} a - \left\lfloor \frac{n}{\sqrt{2}} a \right\rfloor\right)^2 + \left(\frac{n}{\sqrt{2}} b - \left\lfloor \frac{n}{\sqrt{2}} b \right\rfloor\right)^2}$$

  which as above is less than $\frac{2}{n} < \epsilon$ which again is satisfied by choosing $n \in \mathbb{N}$ where $n > \frac{2}{\epsilon}$

(iii) Split up into three parts: $S^{\text{int}} = \varnothing$, $\partial S = [0,1] \times [0,1]$, $\bar{S} = [0,1] \times [0,1]$

(a) As I proved above, every point in S is in the boundary of S, so $S^{\text{int}} = \varnothing$

(b) From above, we have that all points (x,y) where $x, y \in [0,1]$ are rational or irrational (i.e. every point in $[0,1] \times [0,1]$) is in the boundary of S (i.e. that $[0,1] \times [0,1] \subseteq \partial S$). Now if $x < 0$ for some point (x,y), the ball around that point with radius $|x/2|$ is disjoint from S. If $x > 1$, then the ball around that point with radius $|(x-1)/2|$ is disjoint from S. Analogously, all points with $y < 0$ or $y > 1$ have a ball around that point disjoint from S. Hence, $\partial S = [0,1] \times [0,1]$

(c) $\bar{S}$ is defined to be $S \cup \partial S$, so $\bar{S} = [0,1] \times [0,1]$

Problem 2

1. S^{int} is defined as $\{x \in S : B(r,x) \subset S \text{ for some } r > 0\}$. To prove that S^{int} is open using the definition from the book, we must prove that it contains none of its boundary points, i.e. points from $\partial(S^{\text{int}})$. We prove this by proving that all $x \in S^{\text{int}}$ are in $(S^{\text{int}})^{\text{int}}$, which by disjointedness of interiors and boundaries, means that no $x \in S^{\text{int}}$ are boundary points of S^{int}. In math notation, we want to show that for all $x \in S^{\text{int}}$, x is also in $(S^{\text{int}})^{\text{int}} = \{x \in S^{\text{int}} : B(r,x) \subset S^{\text{int}}\}$

For any $x \in S^{\text{int}} \subseteq S$, then by definition, there exists a ball B around x contained in S. For every $y \in B$, because balls are open, I can find $B(r', y) \subset B \subset S$. So for all $y \in B$, y is an interior point of $S \implies y \in S^{\text{int}} \implies B \in S^{\text{int}}$. So tracing the path from the beginning of the paragraph, we have that for any $x \in S^{\text{int}}$, $B(r, x) \subset S^{\text{int}}$, which is exactly what we wanted to prove.

2. The book says that $S^{\text{int}} \cup \partial S \cup (S^C)^{\text{int}}$ is a (disjoint) partition of $\mathbb{R}^n$, and that $\bar{S} = S \cup \partial S = S^{\text{int}} \cup \partial S$ (because S is disjoint from $(S^C)^{\text{int}}$ because S is disjoint from S^C and $(S^C)^{\text{int}} \subseteq S^C$). These two facts imply that $\bar{S} = ((S^C)^{\text{int}})^C$, but the interior of a set is open for any set, so $\bar{S}$ is closed (because S is open $\iff S^C$ is closed)

3. Again using the book's statement that $S^{\text{int}} \cup \partial S \cup (S^C)^{\text{int}}$ is a partition of $\mathbb{R}^n$, we see that $\partial S = (S^{\text{int}} \cup (S^C)^{\text{int}})^C$. From Problem 3 below, the union of two open sets is open, so ∂S is closed.

Problem 3

If S_1, S_2 are open, so $x \in S_1 \implies \exists B(r, x) \subset S_1 \subseteq (S_1 \cup S_2)$, and $x \in S_2 \implies \exists B(r, x) \subset S_2 \subseteq (S_1 \cup S_2)$, so if $x \in S_1 \cup S_2$ then $x \in S_1$ or $x \in S_2$, and from my statements above we see that x is also an interior point of $S_1 \cup S_2 \implies S_1 \cup S_2$ is open.

If $x \in S_1 \cap S_2$, then we see that $\exists B(r_1, x) \subset S_1$ and $\exists B(r_2, x) \subset S_2$, so if we take $r = \min(r_1, r_2)$, $B(r, x)$ is a subset of S_1 and S_2, so x is an interior point of $S_1 \cap S_2 \implies S_1 \cap S_2$ is open.

Problem 4

If S_1, S_2 are closed, then $(S_1)^C$ and $(S_2)^C$ are open. So from Problem 3, we have that $(S_1)^C \cup (S_2)^C = (S_1 \cap S_2)^C$ is open and $(S_1)^C \cap (S_2)^C = (S_1 \cup S_2)^C$ is open. Hence, $S_1 \cap S_2$ is closed and $S_1 \cup S_2$ is closed.

Problem 5

We want to show that $\partial S = \bar{S} \cap \overline{S^C}$. We know from the definition of closure that $\bar{S} = S \cup \partial S$ and $\overline{S^C} = S^C \cup \partial(S^C)$. Because $(S^C)^C = S$, the definition of ∂S is symmetrical across complements, so $\overline{S^C} = S^C \cup \partial S$. From that we get that $\bar{S} \cap \overline{S^C} = (S \cup \partial S) \cap (S^C \cup \partial S)$, but S is disjoint from S^C, so $\bar{S} \cap \overline{S^C} = \partial S$.

Problem 6

$$\bigcup_{k=1}^{\infty} \left[a + \frac{1}{k}, b - \frac{1}{k}\right] = (a, b)$$

because if x is an element of the left hand side, it must be in some $[a + \frac{1}{k}, b - \frac{1}{k}]$ for some finite k, and $[a + \frac{1}{k}, b - \frac{1}{k}] \subset (a, b)$ for every finite k. If x is an element of the right hand side, take $c = \min\{x - a, b - x\}$, and see that $x \in [a + c, b - c] \subset [a + \frac{1}{k}, b - \frac{1}{k}]$ for some finite k.

Problem §1.3 - 3

$$f(x,y) = \frac{\sin(xy)}{x} \text{ for } x \neq 0$$

To make f a continuous function over all of $\mathbb{R}^2$, we need to define $f(0,y)$ for all $y \in \mathbb{R}$ to "fill in the gap". I'll omit the $(x,y) \to (0,y_0)$ under the limit to save space.

$$\lim_{(x,y)\to(0,y_0)} f(x,y) = \lim \frac{y\sin(xy)}{xy} = \lim y \cdot \lim \frac{\sin(xy)}{xy} = y_0 \cdot 1 = y_0$$

because $\lim xy = \lim x \cdot \lim y = 0 \cdot y_0 = 0$, and in class we were given that $\lim_{x\to 0} \frac{\sin x}{x} = 1$. So defining $f(0,y) = y$, the limit approaches the actual value, and hence f is continuous.

Problem 4

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Although f is discontinuous at $(0,0)$ (the book already proved this by approaching along the axes vs. approaching along $y = cx$), $f(x,a)$ and $f(a,y)$ are continuous functions (of x and y respectively) for all $a \in \mathbb{R}$. The proof is rather trivial with the sum, product, and quotient rules for continuous functions:
$f(x,a) = \dfrac{ax}{x^2+a^2}$ for $a \neq 0$ is continuous everywhere because $a \cdot x$ is continuous everywhere and because $x \cdot x + a^2 \neq 0$ is continuous everywhere (which in turn is because of the fact that $y = x$ and $y = a^2$ are continuous everywhere — for $y = x$ choose $\delta = \epsilon$ and for $y = a^2$ choose δ to be literally anything). For $a = 0$, $f(x,0) = 0$ for all $x \neq 0$ and $f(0,0) = (0,0)$ so $f(x,0) = 0$ for all $x \in \mathbb{R}$, which is a continuous function. So all $f(x,a)$ are continuous. $f(a,y)$ is exactly the same due to symmetry.

Problem 5

$$f(x,y) = \begin{cases} \dfrac{y(y-x^2)}{x^4} & \text{if } 0 < y < x^2 \\ 0 & \text{otherwise} \end{cases}$$

First off notice that f is discontinuous at $(0,0)$: approaching along a parabola $y = cx^2$ lying in the region $0 < y < x^2$ (i.e. $c > 1$), we have that

$$\lim_{(x,y)\to(0,0)} \frac{cx^2(cx^2-x^2)}{x^4} = c^2 - c$$

but approaching along the y-axis from above (in the region) and below yields that the limit is $0 \neq c^2 - c$.

Now we just have to prove that there are no other discontinuities. Denote $g(x,y) = \dfrac{y(y-x^2)}{x^4}$. Note that $g(x,y)$ is continuous at every point in the open region $0 < y < x^2$, and the 0 function is continuous at every point in the open regions $y < 0$, and $y > x^2$; hence $f(x,y)$ is continuous on those regions.

[Sidenote]: the fact that they are open regions is important — if they were not open, then at some boundary point every δ-ball would intersect with the region outside of where g and the 0 function are defined (as per the definition of f), which would mean that some (x, y)'s in the ball would not be sent to the right place e.g. $g(x, y)$ or 0, meaning that the conclusion (that f is continuous) might not be valid.

That leaves just the "regions" $y = 0$ and $y = x^2$ where g and the 0 function intersect. Along these curves, g and the 0 function are both 0. Independently, g and the 0 function are continuous at the points of the curves; we just need to prove that when put together, $f(x, y)$ is still continuous at those points. Because those two functions are independently continuous at any one of the points (x_0, y_0) on the curves, that means for every $\epsilon > 0$ $g(x, y)$ has some δ_1 that works and the 0 function has some δ_2 that works. Take $\delta' = \min\{\delta_1, \delta_2\}$. Then for all (x, y) within a δ'-ball of (x_0, y_0), $|f(x, y) - f(x_0, y_0)|$ must be either $|g(x, y) - g(x_0, y_0)| < \epsilon$ or $|0(x, y) - 0(x_0, y_0)| < \epsilon$, and thus, f is continuous everywhere except $(0, 0)$.

Problem 6

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

For $x_0 \neq 0$, $f(x)$ is not continuous at that point: if $x_0 \in \mathbb{Q}$, choose $\epsilon = \dfrac{|x_0|}{2}$. For any δ I try, I can always find an irrational $x' \in (x_0 - \delta, x_0 + \delta)$ (which is always possible because the irrationals are dense, which both I and the book proved) such that $|f(x_0) - f(x')| = |x_0 - 0| = |x_0| \not< \epsilon = \dfrac{|x_0|}{2}$.
If $x_0 \notin \mathbb{Q}$, for $\epsilon = \dfrac{|x_0|}{2}$ and any δ, choose $\delta' = \min\{\epsilon, \delta\}$ and a rational $x' \in (x_0 - \delta', x_0 + \delta') \subseteq (x_0 - \delta, x_0 + \delta)$ (which again we can always do because the rationals are dense). The condition on x' forces that $|x_0 - x'| < \delta$, and then by the reverse triangle inequality, $\big| |x'| - |x_0| \big| \leq |x' - x_0| < \delta' \implies \big| |x'| - |x_0| \big| < \delta' \implies \big| |x_0| - |x'| \big| < \epsilon \implies |x_0| - |x'| < \epsilon \implies |x_0| - \epsilon < |x'| \implies \dfrac{|x_0|}{2} < |x'| \implies \epsilon < |x'|$. Then, $|f(x_0) - f(x')| = |0 - x'| = |x'| \not< \epsilon$.

Now for $x_0 = 0$, for any ϵ, take $\delta = \epsilon$. Then because $f(x) \leq |x|$ for all $x \in \mathbb{R}$, if $|x - x_0| = |x| < \delta$, then $|f(x) - f(x_0)| = |f(x)| \leq |x| < \delta = \epsilon$, so $f(x)$ is continuous at 0.

Problem 7

$$f(x) = \begin{cases} \dfrac{1}{q} & \text{if in simplest form } x = \dfrac{p}{q} \text{ for } p \in \mathbb{Z}, q \in \mathbb{Z}_{\geq 0} \\ 0 & \text{otherwise (i.e. when } x \text{ is irrational)} \end{cases}$$

Similar to above, $f(x)$ is not continuous for non-zero $x_0 \in \mathbb{Q}$ (in fact f is not defined at 0) because for $\epsilon = \frac{1}{2q}$, no matter what δ I choose, there will be some irrational point x' in $(x_0 - \delta, x_0 + \delta)$ (by denseness of irrationals) where $|f(x_0) - f(x')| \not< \epsilon$ (see above, the same exact argument is presented in detail).

For $x_0 \notin \mathbb{Q}$, for any ϵ, I can find some N s.t. $\frac{1}{N} < \epsilon$. Now for all natural numbers $n \leq N$, there is some integer a s.t. $\dfrac{a_n}{n} < x_0 < \dfrac{a_n + 1}{n}$. For every N, you will have $2N$ of these "boundary fractions". Take

$$\delta = \min\left\{\left|x_0 - \frac{a_1}{1}\right|, \left|\frac{a_1+1}{1} - x_0\right|, \left|x_0 - \frac{a_2}{2}\right|, \dots\right\}$$

which we can do because that set only has $2N$ (i.e. finite) elements. Then for any rational number $x' \in (x_0 - \delta, x_0 + \delta)$, it's denominator must be greater than N because all the fractions with denominator N or less are outside of $(x_0 - \delta, x_0 + \delta)$. Hence for all $x' \in (x_0 - \delta, x_0 + \delta)$, $|f(x') - f(x_0)| < \frac{1}{N} < \epsilon$.

Problem §1.4 - 1

(a) $x_k = \dfrac{\sqrt{2k+1}}{2\sqrt{k}+1}$. We can divide top and bottom by $\sqrt{k}$ to get $x_k = \dfrac{\frac{\sqrt{2k+1}}{\sqrt{k}}}{\frac{2\sqrt{k}+1}{\sqrt{k}}} = \dfrac{\sqrt{2+\frac{1}{k}}}{2+\frac{1}{\sqrt{k}}}$. As $k \to \infty$, $\frac{1}{k} \to 0$ and $\frac{1}{\sqrt{k}} \to 0$, so $\lim_{k\to\infty} \frac{\sqrt{2+\frac{1}{k}}}{2+\frac{1}{\sqrt{k}}} = \frac{\sqrt{2}}{2}$

(b) $x_k = \frac{\sin k}{k}$. Because $|\sin x| \leq 1$, we have that $x_k \leq \frac{1}{k}$, which converges to 0. Hence $x_k \to 0$.

(c) $x_k = \sin\frac{k\pi}{3}$ diverges: for any $N \in \mathbb{N}$, pick $\epsilon = \frac{1}{2}$. I will always be able to find $k_0 \equiv 0 \pmod 6$ and $k_1 \equiv 1 \pmod 6$ where $|x_{k_1} - x_{k_0}| = \left|\sin\dfrac{\pi}{3} - \sin 0\right| = \frac{\sqrt{3}}{2} \not< \frac{1}{2} = \epsilon$

Problem 6

We want to prove that a is an accumulation point of a set $S \iff$ there is a sequence of $x_k \in S, x_k \neq a$ that tends to a.

($\implies$) If a is an accumulation point, then by definition every open ball around a has infinitely many points from S. To create a convergent sequence of $x_k \to a$, just set x_k to be one of the infinitely many points in the intersection of the open ball $B(\frac{1}{k}, a)$ and S. (Convergent because for $k > \frac{1}{\epsilon}$, $|x_k - a| < \frac{1}{k} < \epsilon$). [Axiom of Choice? Hmmm.]

($\impliedby$) If there is a convergent sequence $x_k \to a$, that means that given an ϵ, all the x_k past a certain point will be within ϵ of a. Hence for any ϵ-ball, the convergent sequence gives infinitely many points contained in both S and the ϵ-ball $\implies$ a is an accumulation point.

Problem 7

We want to prove that $\bar{S} = S \cup \partial S = S \cup A$ where A is the set of all S's accumulation points:

($\subseteq$) Suppose there is a point $x \in \bar{S} \setminus S$ that's not in A. That means that there must be some open ball B around x s.t. there are only finitely many points of S in B. Take the minimum of all distances between this finite set of points and x, and half it. Then clearly a ball of that radius around x does not intersect with S, which means x is not in S or ∂S, contradicting our assumption that $x \in \bar{S}$.

($\supseteq$) If a point is in S, then obviously it's in $S \cup \partial S$. If a point is in A, then every open ball has infinitely many points in the intersection between itself and S. The point can't possibly be $\bar{S}^C = (S^C)^{\text{int}}$ (this equality is proven in Problem 2 from §1.2) because all points in $(S^C)^{\text{int}}$ have open balls contained completely inside S^C. Hence, the point must be in $\bar{S}$.