Primes and Bertrand's Postulate

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Bertrand's postulate states that there must be at least one prime between any positive n and $2n$.

1 Necessary Functions

The Von-Mangoldt function $\Lambda(n)$ is defined as

$$\Lambda(n) = \begin{cases} \ln p & \text{if } n = p^k \\ 0 & \text{for all other cases} \end{cases}$$

$\Lambda(n)$ for the first 10 numbers is

$$0, \ln 2, \ln 3, \ln 2, \ln 5, 0, \ln 7, \ln 2, \ln 3, 0$$

The first Chebyshev function $\vartheta(x)$ is defined as

$$\vartheta(x) = \sum_{p \leq x} \ln(p) \tag{1}$$

The first few values are

$$0, \ln 2, \ln 6, \ln 6, \ln 30, \ln 30, \ln 210, \ln 210, \ln 210, \ln 210$$

The second Chebyshev function $\psi(x)$ is defined as

$$\psi(x) = \sum_{n \leq x} \Lambda(n) = \sum_{p^k \leq x} \ln p = \sum_{i \geq 1} \sum_{p^i \leq x} \ln p$$

The first few values are

$$0, \ln 2, \ln 6, \ln 12, \ln 60, \ln 60, \ln 420, \ln 840, \ln 2920, \ln 2920$$

Also, note that

$$\ln n = \sum_{d|n} \Lambda(d)$$

where $d|n$ means all d that divide n, or all the divisors of n, which makes sense because this is essentially just looking through the divisors of n, finding the prime powers, and multiplying them, aka factoring n. The second Chebyshev function can also be represented like so

$$\psi(x) = \ln(lcm(\text{all the integers from } 1 \ldots x))$$

because every time there is a new prime factor, the von Mangoldt function makes note of it and the second Chebyshev function multiplies it with the lcm value it already has, but repeated prime factors are 0 for the von-Mangoldt function and thus are not seen in $\psi(x)$. Cheb1 and Cheb2 look really similar, but there is no obvious connection between them. However, as is with most mathematics, if you dig deep enough, there will always be a connection

$$\begin{aligned} \exp\left(\sum_{i \geq 1} \vartheta(x^{1/i})\right) &= \prod_{i \geq 1} \exp\left(\vartheta(x^{1/i})\right) = \prod_{i \geq 1} \prod_{p \leq x^{1/i}} p \\ &= \prod_{i \geq 1} \prod_{p^i \leq x} p = \exp\left(\ln\left(\prod_{i \geq 1} \prod_{p^i \leq x} p\right)\right) = \exp\left(\sum_{i \geq 1} \sum_{p^i \leq x} \ln p\right) = e^{\psi(x)} \end{aligned}$$

which means that

$$\sum_{i\geq 1} \vartheta(x^{1/i}) = \psi(x) \tag{2}$$

Another important identity (just using above definitions and manipulating sums) is as follows

$$\begin{aligned}
\ln(\lfloor x \rfloor !) &= \sum_{n\leq x} \ln(n) = \sum_{n\leq x} \sum_{d|n} \Lambda(d) \\
&= \sum_{dk\leq x} \Lambda(d) = \sum_{k\leq x} \sum_{d\leq x/k} \Lambda(d) \\
&= \sum_{1\leq k\leq x} \psi(x/k) = \sum_{k\geq 1} \psi(x/k).
\end{aligned}$$

and condensing

$$\ln(\lfloor x \rfloor !) = \sum_{k\geq 1} \psi(x/k) \tag{3}$$

2 Setting Bounds

From equation (2), we get

$$\psi(x) = \sum_{i\geq 1} \vartheta(x^{1/i})$$

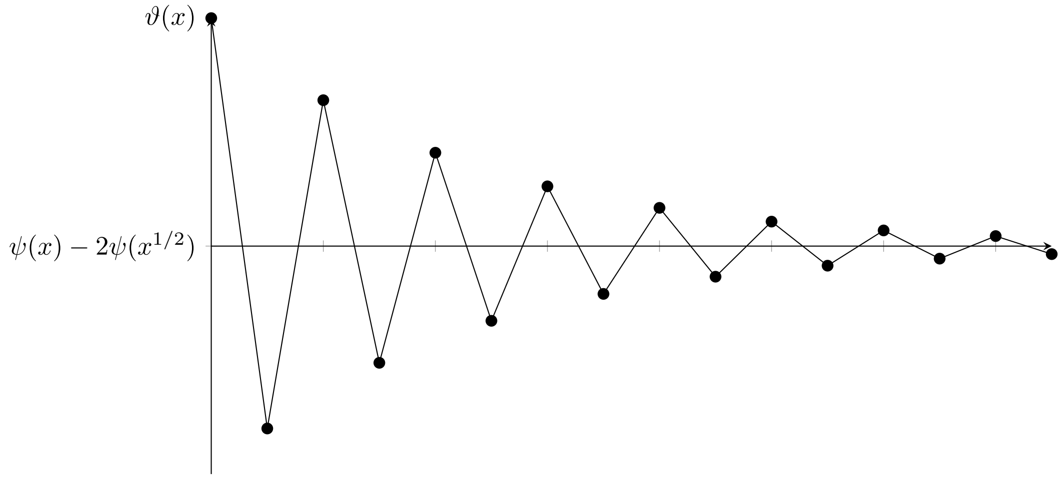
and

$$2\psi(x^{1/2}) = 2\sum_{i\geq 1} \vartheta(x^{1/2i})$$

Subtracting the latter, all the even numbered terms turn negative, giving us

$$\psi(x) - 2\psi(x^{1/2}) = \vartheta(x) - \vartheta(\sqrt{x}) + \vartheta(\sqrt[3]{x}) - \dots$$

Visualizing this, we get something like this:



Looking at the diagram, it's obvious that $\vartheta(x)$ is greater than $\psi(x) - 2\psi(x^{1/2})$, and $\psi(x)$ is greater than $\vartheta(x)$ by the definition, so

$$\psi(x) \geq \vartheta(x) \geq \psi(x) - 2\psi(x^{1/2}) \tag{4}$$

By similar reasoning with alternating plots (and from identity (3)), we get that

$$\psi(x) - \psi\left(\frac{x}{2}\right) \leq \ln(\lfloor x \rfloor !) - 2\ln\left(\left\lfloor \frac{x}{2} \right\rfloor !\right) \leq \psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right) \tag{5}$$

We know that

$$x! = \Gamma(x+1) = x\Gamma(x)$$

It is quite easy to see that

$$\lfloor x \rfloor ! \geq \Gamma(x)$$

because with $x > 1$, it is obvious, and for $x < 1$, the floor function makes the $x! = 1$, while $\Gamma(x < 1) \leq 1$. Likewise,for $x > 1$,

$$\lfloor x \rfloor ! \leq \Gamma(x+1)$$

and

$$\left\lfloor \frac{x}{2} - \frac{1}{2} \right\rfloor ! \leq \Gamma\left(\frac{1}{2}x + \frac{1}{2}\right)$$

With these inequalities, we can achieve that for $x > 2$

$$\ln \Gamma(x+1) - 2\ln \Gamma\left(\frac{1}{2}x + \frac{1}{2}\right) \geq \ln(\lfloor x \rfloor !) - 2\ln\left(\left\lfloor \frac{x}{2} \right\rfloor !\right) \geq \ln \Gamma(x) - 2\ln \Gamma\left(\frac{1}{2}x + \frac{1}{2}\right)$$

as pictured below:

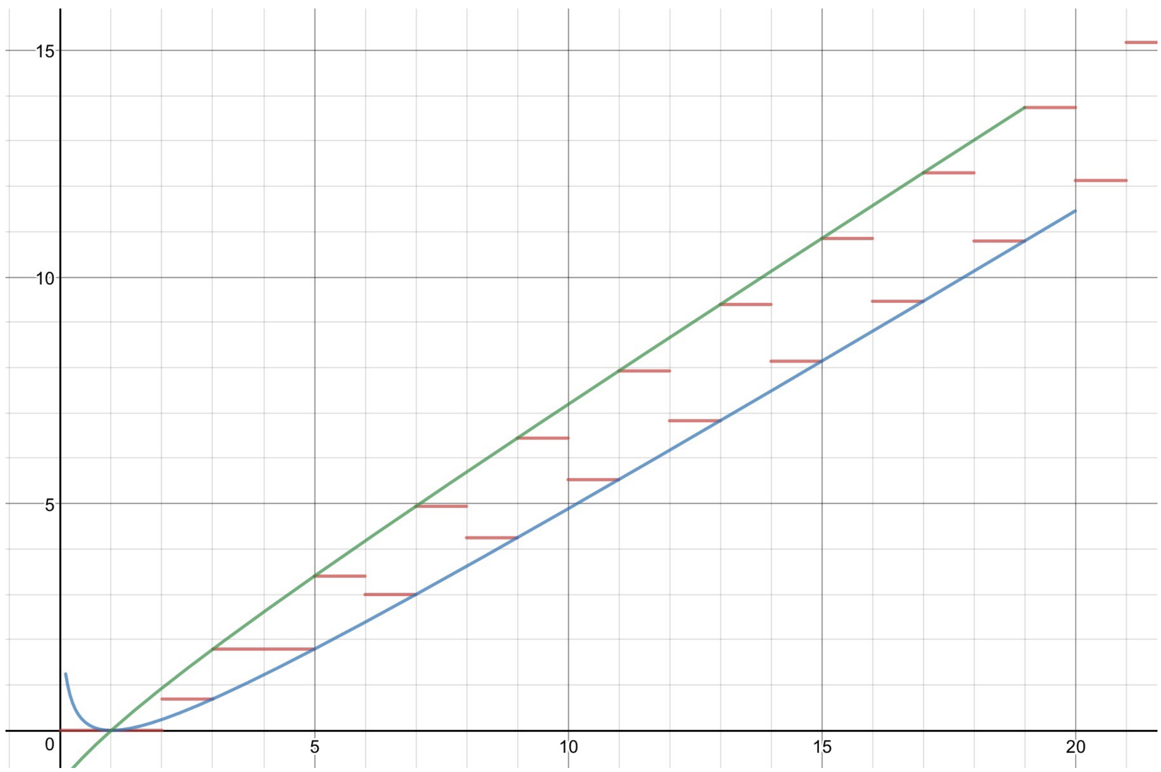


Figure 1: ln(Gamma) Inequalities

3 Stirling's Approximation

Now that we have our inequalities all listed out, we can use Stirling's Aprroximation. Recall that

$$n\Gamma(n) = n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Thus

$$\Gamma(n) = \sqrt{\frac{2\pi}{n}}\left(\frac{n}{e}\right)^n$$

Using it in our first inequality,

$$\frac{1}{2}\ln\left(\frac{2\pi}{n+1}\right) + (n+1)\ln\left(\frac{n+1}{e}\right) - \ln\left(\frac{4\pi}{n+1}\right) - (n+1)\ln\left(\frac{n+1}{2e}\right) \geq \ln(\lfloor n\rfloor!) - 2\ln\left(\left\lfloor\frac{n}{2}\right\rfloor!\right)$$

which is just

$$\frac{1}{2}\ln\left(\frac{n+1}{8\pi}\right) + (n+1)\ln 2 \geq \ln(\lfloor n\rfloor!) - 2\ln\left(\left\lfloor\frac{n}{2}\right\rfloor!\right)$$

Using it in our second inequality,

$$\frac{1}{2}\ln\left(\frac{2\pi}{n}\right) + n\ln\left(\frac{n}{e}\right) - \ln\left(\frac{4\pi}{n+1}\right) - (n+1)\ln\left(\frac{n+1}{2e}\right) \leq \ln(\lfloor n\rfloor!) - 2\ln\left(\left\lfloor\frac{n}{2}\right\rfloor!\right)$$

which is just

$$\frac{1}{2}\ln\left(\frac{(n+1)^2}{8\pi n}\right) + (n+1)\ln\left(\frac{2n}{n+1}\right) - \ln\left(\frac{n}{e}\right) \leq \ln(\lfloor n\rfloor!) - 2\ln\left(\left\lfloor\frac{n}{2}\right\rfloor!\right)$$

These two new bounds are easier to work with as they are just logarithmic functions compared to Gamma functions; a bit of luck (or some graphing and deriving to ensure the difference the linear and logarithmic bounds is increasing and thus will always enclose $\ln(\lfloor n\rfloor!) - 2\ln\left(\left\lfloor\frac{n}{2}\right\rfloor!\right)$) will reveal that for big enough x,

$$\frac{2}{3}x < \ln(\lfloor n\rfloor!) - 2\ln\left(\left\lfloor\frac{n}{2}\right\rfloor!\right) < \frac{3}{4}x \tag{6}$$

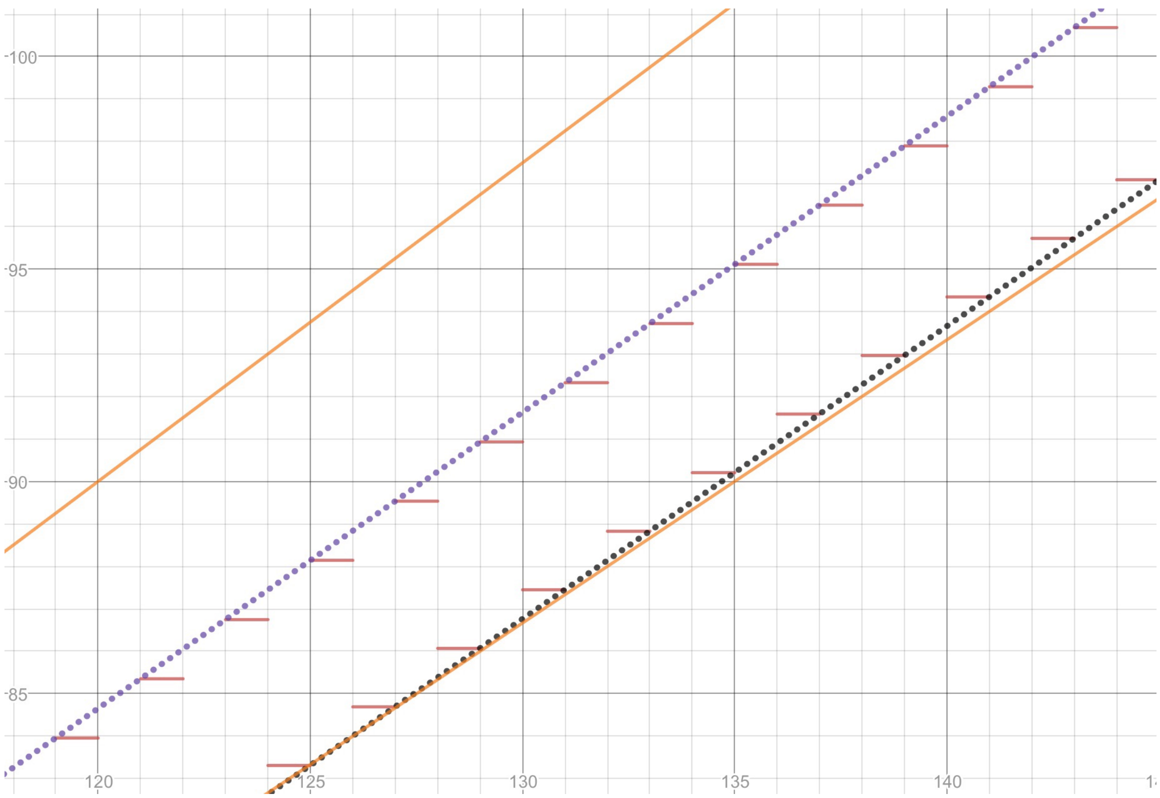


Figure 2: Linear bounds of logarithmic bounds

4 Finale

From (5),

$$\psi(x) - \psi\left(\frac{x}{2}\right) \leq \ln(\lfloor x \rfloor !) - 2\ln\left(\left\lfloor \frac{x}{2} \right\rfloor !\right) \leq \psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right)$$

and from (6),

$$\frac{2}{3}x < \ln(\lfloor n \rfloor !) - 2\ln\left(\left\lfloor \frac{n}{2} \right\rfloor !\right) < \frac{3}{4}x$$

which means that

$$\psi(x) - \psi\left(\frac{x}{2}\right) < \frac{3}{4}x, \text{ and for } x \text{ greater than say 200, } \frac{2}{3}x < \psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right) \tag{7}$$

Summing the first inquality over x values that half each time yields

$$\psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{2}\right) - \psi\left(\frac{x}{4}\right) + \psi\left(\frac{x}{4}\right) - \psi\left(\frac{x}{8}\right) + \ldots < \frac{3}{4}x + \frac{3}{8}x + \frac{3}{16} + \ldots$$

which means

$$\psi(x) < \frac{3}{2}x \text{ for } x > 0 \tag{8}$$

From (4),

$$\psi(x) \geq \vartheta(x) \geq \psi(x) - 2\psi(x^{1/2})$$

Thus, we have

$$\psi(x) \leq \vartheta(x) + 2\psi(\sqrt{x})$$
$$-\psi\left(\frac{x}{2}\right) \leq -\vartheta\left(\frac{x}{2}\right)$$
$$\psi\left(\frac{x}{3}\right) = \psi\left(\frac{x}{3}\right)$$

Adding everything, we get that

$$\psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right) \leq \vartheta(x) + 2\psi(\sqrt{x}) - \vartheta\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right)$$

Using (8), we can simplify the inequality to be

$$\psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right) < \vartheta(x) - \vartheta\left(\frac{x}{2}\right) + \frac{x}{2} + 3\sqrt{x}$$

For large x, we can use the second inequality of (7)

$$\frac{2}{3}x < \vartheta(x) - \vartheta\left(\frac{x}{2}\right) + \frac{x}{2} + 3\sqrt{x}$$

which yields

$$\vartheta(x) - \vartheta(x/2) > \frac{1}{6}x - 3\sqrt{x} \tag{9}$$

And from basic algebra and/or graphing

$$\frac{1}{6}x - 3\sqrt{x} \geq 0, \text{ if } x \geq 324, \text{ which is in fact large enough}$$

we know that for $x = 2n$ $n \geq 162$,

$$\vartheta(2n) - \vartheta(n) > 0 \quad \square$$

WHICH MEANS THERE MUST BE A PRIME IN BETWEEN n AND $2n$ for $n \geq 162$!!! We can very easily verify that for $n < 162$, there is indeed a prime between n and $2n$, thus proving once and for all Bertrand's Postulate!