The Cubic Formula and Derivation

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Here is the general cubic, with the x^3 coefficient already divided into the other coefficients, right hand side already set to zero because we are finding roots: $x^3 + ax^2 + bx + c = 0$.

We substitute in $x = y - \frac{a}{3}$ to get

$$\begin{aligned} \left(y^3 - \frac{2a}{3}y^2 + \frac{a^2}{9}y - \frac{a}{3}y^2 + \frac{2a^2}{9}y - \frac{a^3}{27}\right) \\ +a\left(y^2 - \frac{2a}{3}y + \frac{a^2}{9}\right) \\ +b\left(y - \frac{a}{3}\right) \\ +c = 0 \end{aligned}$$

Simplifying to give

$$\begin{aligned} y^3 - ay^2 + \frac{a^2}{3}y - \frac{a^3}{27} \\ +ay^2 - \frac{2a^2}{3}y + \frac{a^3}{9} \\ +by - \frac{ab}{3} \\ +c = 0 \end{aligned}$$

Simplifying more to give

$$y^3 - \frac{a^2}{3}y + by - \frac{a^3}{27} + \frac{a^3}{9} - \frac{ab}{3} + c = 0$$

This is of the form $y^3 + dy + e = 0$ where

$$d = b - \frac{a^2}{3} \text{ and } e = -\frac{a^3}{27} + \frac{a^3}{9} - \frac{ab}{3} + c$$

We substitute $y = \sqrt[3]{u} - \sqrt[3]{v}$ to get

$$\sqrt[3]{u}^3 - 3\sqrt[3]{u}^2\sqrt[3]{v} + 3\sqrt[3]{u}\sqrt[3]{v}^2 - \sqrt[3]{v}^3 + d\sqrt[3]{u} - d\sqrt[3]{v} + e = 0$$

Simplifying to get

$$(u - v) - d\sqrt[3]{v} - 3\sqrt[3]{u}^2\sqrt[3]{v} + d\sqrt[3]{u} + 3\sqrt[3]{u}\sqrt[3]{v}^2 + e = 0$$

Anti-distributing to get

$$(u - v) - \sqrt[3]{v}(d - 3\sqrt[3]{u}^2) + \sqrt[3]{u}(d + 3\sqrt[3]{v}^2) + e = 0$$

Let's define $v - u = e$ ('cause we can), causing $-\sqrt[3]{v}(d - 3\sqrt[3]{u}^2) + \sqrt[3]{u}(d + 3\sqrt[3]{v}^2) = 0$, which can be simplified to

$$\sqrt[3]{u}(d + 3\sqrt[3]{v}^2) = \sqrt[3]{v}(d + 3\sqrt[3]{u}^2)$$

$$d\sqrt[3]{u} + 3\sqrt[3]{u}\sqrt[3]{v}^2 = d\sqrt[3]{v} + 3\sqrt[3]{v}\sqrt[3]{u}^2$$

$$d\sqrt[3]{u} - d\sqrt[3]{v} = 3\sqrt[3]{v}\sqrt[3]{u}^2 - 3\sqrt[3]{u}\sqrt[3]{v}^2$$

$$d(\sqrt[3]{u} - \sqrt[3]{v}) = 3\sqrt[3]{v}\sqrt[3]{u}(\sqrt[3]{u} - \sqrt[3]{v})$$

$$d = 3\sqrt[3]{v}\sqrt[3]{u}$$

$$\frac{d}{3} = \sqrt[3]{uv}$$

$$\frac{d^3}{27} = uv$$

From the first purple equation, we have $v = e + u$, which we can put into the second to get

$$u(e + u) = \frac{d^3}{27} \to u^2 + eu - \frac{d^3}{27} = 0$$

Where we can solve for u with the quadratic formula.

$$u = \frac{-e \pm \sqrt{e^2 + 4\frac{d^3}{27}}}{2}$$

We can also get equations for v; the first equation gives $u = v - e$, which we stuff into the second equation to get

$$v(v-e) = \frac{d^3}{27} \rightarrow v^2 - ev - \frac{d^3}{27}$$

Quadratic formula yielding

$$v = \frac{e \pm \sqrt{e^2 + 4\frac{d^3}{27}}}{2}$$

Via our definition above ($y = \sqrt[3]{u} - \sqrt[3]{v}$), we get

$$y = \sqrt[3]{\frac{-e \pm \sqrt{e^2 + 4\frac{d^3}{27}}}{2}} - \sqrt[3]{\frac{e \pm \sqrt{e^2 + 4\frac{d^3}{27}}}{2}}$$

And from the definition above that ($x = y - \frac{a}{3}$), we have

$$x = \sqrt[3]{\frac{-e \pm \sqrt{e^2 + 4\frac{d^3}{27}}}{2}} - \sqrt[3]{\frac{e \pm \sqrt{e^2 + 4\frac{d^3}{27}}}{2}} - \frac{a}{3}$$

However, there is a problem. The $\pm$ gives solutions that don't satisfy $e = v - u$, so we just keep the positive. And finally, if we want, we can plug in d and e to get the cubic formula

$$x = \sqrt[3]{\frac{-(-\frac{a^3}{27} + \frac{a^3}{9} - \frac{ab}{3} + c) + \sqrt{(-\frac{a^3}{27} + \frac{a^3}{9} - \frac{ab}{3} + c)^2 + 4\frac{(b-\frac{a^2}{3})^3}{27}}}{2}}$$
$$- \sqrt[3]{\frac{(-\frac{a^3}{27} + \frac{a^3}{9} - \frac{ab}{3} + c) + \sqrt{(-\frac{a^3}{27} + \frac{a^3}{9} - \frac{ab}{3} + c)^2 + 4\frac{(b-\frac{a^2}{3})^3}{27}}}{2}} - \frac{a}{3}$$

This formula only gives one root; using roots of unity we can get the others.