On Differentiability

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Functions of a Single Variable

In the single variable case, the definition of differentiability (at a point a) is that of the limiting slope of a secant/tangent line at a:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = m$$

where m is then defined as the derivative of f at a, or in other words, for all $\epsilon > 0$, there is $\delta > 0$ s.t.

$$|x - a| < \delta \implies \left| \frac{f(x) - f(a)}{x - a} - m \right| < \epsilon$$

or if we define $g(x) = f(x) - L(x)$ where $L(x) = f(a) + m(x - a)$ (the linear approximation to f at a), then the above is equivalent to

$$|x - a| < \delta \implies \left| \frac{g(x)}{x - a} \right| < \epsilon \iff |g(x)| < \epsilon |x - a|$$

A Generalization

Now in higher dimensions, having $(x - a)$ in the denominator is no good, because we can't divide by a vector. However, we can tweak the above definition by noticing that

$$\left| \frac{g(x)}{x - a} \right| < \epsilon \iff |g(x)| < \epsilon |x - a|$$

to get rid of that problem. Unfortunately, g no longer makes sense as $f(x) - (f(a) + m(x - a))$, but we can generalize that easily as $g(x) = f(x) - L(x)$ where L is some linear function (e.g. a line, plane, hyperplane, etc). So we can now rephrase the above single-variable definition for multiple variables: a function $f : \mathbb{R}^m \to \mathbb{R}$ is differentiable at a point $\mathbf{a} \in \mathbb{R}^m$ if there is some linear function $L : \mathbb{R}^m \to \mathbb{R}$ (again, the linear approximation to f at $\mathbf{a}$), defined as $L(x) = f(\mathbf{a}) + D(\mathbf{x} - \mathbf{a})$ where D is a linear transformation that takes $\mathbb{R}^m$ vectors to $\mathbb{R}$ vectors (i.e. it is a 1 by m matrix) s.t. for $g = f - L$, for every ϵ we can find δ s.t.

$$|\mathbf{x} - \mathbf{a}| < \delta \implies |g(\mathbf{x})| < \epsilon |\mathbf{x} - \mathbf{a}|$$

In English, we can interpret this condition as: around a neighborhood of $\mathbf{a}$, if we subtract the linear approximation to f at $\mathbf{a}$ from f, then that function is below any "cone" (because $|\mathbf{x} - \mathbf{a}|$ is a "cone", at least in 1- and 2-dimensions), which intuitively/geometrically we see as being "smooth". Alternatively, we can think the condition as saying that g goes to 0 near $\mathbf{a}$ "faster" than any linear function. And as above where we thought of m as the derivative for that function at that point, we can think of D as the "derivative" of f at $\mathbf{a}$.

Differentiability based on Continuous Functions

In class, Jim gave another definition of continuity for functions $f : \mathbb{R}^m \to \mathbb{R}$: if there exists a continuous function $p(\mathbf{x}) : \mathbb{R}^m \to \mathbb{R}^m$ s.t. $f(\mathbf{x}) - f(\mathbf{a}) = p(\mathbf{x}) \cdot (\mathbf{x} - \mathbf{a})$ where we can define $p(\mathbf{a})$ is the "derivative".

Equivalence

PROVE THAT BOTH DEFINITIONS ARE EQUIVALENT (and explain your solution to my satisfaction) FOR 1 DOLLAR FROM DANIEL RUI.