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Chapter 1

1.1 The Gamma Function

Reminders: We can prove quite easily (using integraton by parts) that: $\Gamma(x+1) = x\Gamma(x)$, which is a fundamental property of the factorial function. The gamma function is defined to be:

$$\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}\,dt = (x-1)! \text{ for positive integers } x$$

The whole purpose of the Gamma Function was to continuously define the factorial function for all numbers (we'll ignore the non-continuous negative half in this paper). Therefore, we should be able to find the "factorial" of a half, also known as $\Gamma\left(\frac{3}{2}\right)$, but as you may have guessed, the integral

$$\int_0^\infty \sqrt{t}e^{-t}\,dt$$

is not a very easy integral to solve! And guess what we do in this paper? We do it anyways.

1.2 The Reflection Formula

Let's begin with another introduction. This is the gamma reflection formula:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

With the gamma reflection formula, solving for $\frac{1}{2}!$ is really easy. Set $z = \frac{1}{2}$ to get:

$$\Gamma\left(\frac{1}{2}\right)\Gamma\left(1-\frac{1}{2}\right) = \frac{\pi}{\sin(\frac{\pi}{2})}$$

so $\Gamma(\frac{1}{2})^2 = \pi$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, and because $\Gamma(x+1) = x\Gamma(x)$, $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$, so

$$\frac{1}{2}! = \frac{\sqrt{\pi}}{2} \quad \square$$

Now that we have established the fact that we can find one half factorial from the reflection formula, we set out to prove it.

1.2.1 Integration

First things first, we plug in variables into the definition of the gamma function, in this case t and s, to allow combining the integrals together. This works because the s integral is a 'constant' in the eyes of the t integral, and constants can be pulled inside, allowing us to simplify the two integrals into just one integral with two variables

$$\begin{aligned}
\Gamma(z)\Gamma(1-z) &= \int_0^\infty s^{z-1}e^{-s}\,ds \int_0^\infty t^{(1-z)-1}e^{-t}\,dt \\
&= \int_0^\infty \Big(\int_0^\infty s^{z-1}e^{-s}\,ds\Big)\, t^{-z}e^{-t}\,dt \\
&= \int_0^\infty \int_0^\infty s^{z-1}e^{-s}t^{-z}e^{-t}\,ds\,dt \\
&= \int_0^\infty \int_0^\infty \frac{s^{z-1}}{t^z}e^{-(s+t)}\,ds\,dt \\
&= \int_0^\infty \int_0^\infty \frac{s^{z-1}}{t^{z-1}}\frac{1}{t}e^{-(s+t)}\,ds\,dt
\end{aligned}$$

We do a variable change $u = s + t$ and $v = \frac{s}{t}$. Notice that $1 + v = \frac{u}{t}$, so $\frac{1+v}{u} = \frac{1}{t}$.

$$\int_0^\infty \int_0^\infty \frac{v^{z-1}}{u}(1+v)e^{-u}\,ds\,dt$$

In order to go from $ds\,dt$ to $du\,dv$ however,we have to do something fancier: the Jacobian

$$du\,dv = \det\begin{bmatrix} \frac{\partial u}{\partial t} & \frac{\partial v}{\partial t} \\ \frac{\partial u}{\partial s} & \frac{\partial v}{\partial s} \end{bmatrix} ds\,dt$$

Where 'det' means determinant and ∂ is just the partial derivative. For more info on Jacobians, visit https://www.quora.com/What-is-an-intuitive-explanation-of-Jacobians-and-a-change-of-basis. Filling the matrix out, we get

$$du\,dv = \det\begin{bmatrix} 1 & -\frac{s}{t^2} \\ 1 & \frac{1}{t} \end{bmatrix} ds\,dt$$

Calculating, we get

$$\begin{aligned}
du\,dv &= \frac{1}{t} + \frac{s}{t^2}ds\,dt \\
&= \frac{t+s}{t^2}ds\,dt \\
&= u\frac{(1+v)^2}{u^2}ds\,dt \\
&= \frac{(1+v)^2}{u}ds\,dt
\end{aligned}$$

Rearranging the equation to be more useful,

$$\frac{1}{1+v}du\,dv = \frac{1+v}{u}ds\,dt$$

and substituting back, we get

$$\int_0^\infty \int_0^\infty \frac{v^{z-1}}{1+v}e^{-u}\,du\,dv$$

Now that we have everything in u and v, we can pull the the integrals apart, the same way we originally combined them

$$\int_0^\infty \frac{v^{z-1}}{1+v}\,dv \int_0^\infty e^{-u}\,du$$

The u integral tells us that it goes to 1 (verify on your own), so all we have left is our v integral:

$$\int_0^\infty \frac{v^{z-1}}{1+v}\,dv$$

This integral is not a particularly easy integral to evaluate, so we split it up into two integrals:

$$\int_0^1 \frac{v^{z-1}}{1+v}\,dv + \int_1^\infty \frac{v^{z-1}}{1+v}\,dv$$

We then do b-substitution on the second integral where $v = \frac{1}{b}$ (so $dv = -\frac{1}{b^2}\,db$ and the bounds from 1 to 0), we get:

$$\int_1^0 \frac{b^{1-z}}{\frac{b+1}{b}}\,\frac{-1}{b^2}\,db = \int_1^0 \frac{b^{2-z}}{b+1}\,\frac{-1}{b^2}\,db = \int_0^1 \frac{b^{-z}}{b+1}\,db$$

Now, we can rename b to v (yes, we can do that), to get:

$$\int_0^1 \frac{v^{z-1}}{1+v}\,dv + \int_0^1 \frac{v^{-z}}{v+1}\,dv = \int_0^1 \frac{v^{z-1}+v^{-z}}{1+v}\,dv$$

But $\frac{1}{1+v}$ can be represented as an infinite geometric series: $\sum_{n=0}^{\infty}(-v)^n$, and replacing the fraction with the sum, we get:

$$\int_0^1 (v^{z-1}+v^{-z})\sum_{n=0}^{\infty}(-v)^n\,dv$$

In the sum's eyes, $v^{z-1}+v^{-z}$ is a constant because it doesn't have any n's in it, so we can pull it inside and rearrange (sum of integrals = integral of sum)

$$\int_0^1 \sum_{n=0}^{\infty}(v^{z-1}+v^{-z})(v)^n(-1)^n\,dv = \sum_{n=0}^{\infty}\int_0^1 (v^{z-1}+v^{-z})(v)^n(-1)^n\,dv$$

In the integral's eyes, the $(-1)^n$ is a constant because it doesn't have any v's

$$\sum_{n=0}^{\infty}(-1)^n \int_0^1 (v^{z-1}+v^{-z})\,v^n\,dv$$

And distributing the v^n

$$\sum_{n=0}^{\infty}(-1)^n \int_0^1 (v^{n+z-1} + v^{-z+n})\, dv$$

then integrating (power rule)

$$\sum_{n=0}^{\infty}(-1)^n \left(\frac{v^{n+z}}{n+z} + \frac{v^{-z+n+1}}{-z+n+1}\right)\Bigg|_{v=0}^{v=1} = \sum_{n=0}^{\infty}(-1)^n \left(\frac{1}{n+z} + \frac{1}{n+1-z}\right)$$

and then writing it out and shrinking it again (verify on your own), we get

$$\frac{1}{z} + \sum_{n=1}^{\infty}(-1)^n \left(\frac{1}{n+z} + \frac{1}{n-z}\right)$$

Of course, this is yet again another problem we can't easily solve. We now take a step back, and look for another way to approach the problem: trig!

1.3 Fourier Series

Fourier Series are all about representing any periodic function in the form $\sum_{n=0}^{\infty} a_n \cos(nx)$. Let's start with the periodic function $\cos(zx)$, where $z \in \Re$ but not necessarily an integer.

1.3.1 Trig Integration

Note: all following integrals without a dx should have a dx.
Trig flashback: recall that

$$\cos^2(x) + \sin^2(x) = 1$$

and

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

Setting $\alpha = \beta = x$, we get

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

Adding the first equation and the one right above this sentence, we get

$$2\cos^2(x) = 1 + \cos(2x) \implies \cos^2(x) = \frac{\cos(2x) + 1}{2}$$

Keep this in mind as we keep going. Next, we must evaluate the tricky integral (the Fourier series)

$$\int_0^{\pi} \cos(zx)\cos(nx)\, dx$$

Integration by parts (verify on your own) gives

$$\left. \frac{\frac{1}{n}\cos(zx)\sin(nx) - \frac{z}{n^2}\cos(nx)\sin(zx)}{1-\frac{z^2}{n^2}} \right|_{x=0}^{x=\pi}$$

If n and z are integers and $n \neq z$, then everything is zero because $\sin(k\pi) = 0$ for all integers k.
If $n = z$ and n and z are integers, we get

$$\int_0^\pi cos^2(nx)\,dx = \frac{1}{2}\int_0^\pi cos(2nx) + 1\,dx$$

$$= \frac{1}{2}\left(\frac{1}{2}sin(2nx) + x\right)\Big|_0^\pi = \frac{\pi}{2}$$

If n is an integer and z not an integer, $\sin(n\pi)$ will always be 0, and $\cos(n\pi)$ will go back and forth between -1 and 1, so we can replace it with $(-1)^n$

$$-(-1)^n \frac{z\sin(z\pi)}{n^2 - z^2}$$

Splitting $\frac{1}{n^2-z^2}$ into two fractions, we get $\frac{1}{2z(n-z)} - \frac{1}{2z(z+n)}$, and because $n - z = -(z-n)$, we can simplify

$$-(-1)^n z\,\sin(z\pi)\left(-\frac{1}{2z(z-n)} - \frac{1}{2z(z+n)}\right) = \frac{1}{2}(-1)^n\sin(z\pi)\left(\frac{1}{z-n} + \frac{1}{z+n}\right)$$

To find the Fourier Series of $cos(zx)$ we need to find a_n's such that

$$\cos(zx) := a_0\,\cos(0x) + a_1\,\cos(x) + a_2\,\cos(2x) + \ldots + a_n\,\cos(nx) + \ldots$$

First, let's find a_0. Take $\cos(zx) := a_0\,\cos(0x) + a_1\,\cos(x) + a_2\,\cos(2x) + \ldots + a_n\,\cos(nx) + \ldots$, and integrate it from 0 to π.

$$\int_0^\pi \cos(zx) = \int_0^\pi a_0\,\cos(0x) + \int_0^\pi a_1\,\cos(x) + \ldots + \int_0^\pi a_n\,\cos(nx) + \ldots$$

but what is $\int_0^\pi \cos(nx)$ where n is an integer but not 0? It's just 0! (Think graphically; all the peaks cancel out the troughs)

$$\int_0^\pi \cos(zx) = \int_0^\pi a_0\,dx \implies \frac{1}{z}\sin(zx)\Big|_0^\pi = \pi a_0 \implies a_0 = \frac{1}{z\pi}sin(z\pi)$$

Next, let's find a_n. Multiplying everything by $\cos(nx)$,

$$\cos(zx)\cos(nx) = a_0\,\cos(nx) + a_1\,\cos(x)\cos(nx) + \ldots + a_n\,\cos(nx)\cos(nx) + \ldots$$

And then integrating from 0 to π

$$\int_0^\pi \cos(zx)\cos(nx) = \int_0^\pi a_0\ \cos(nx) + \int_0^\pi a_1\ \cos(x)\cos(nx) + \ldots + \int_0^\pi a_n\ \cos(nx)\cos(nx) + \ldots$$

But remember the red stuff above? All the $\cos(\text{integer}x)\cos(nx)$ are all zero!

$$\int_0^\pi \cos(zx)\cos(nx) = 0 + 0 + \ldots + \int_0^\pi a_n\ \cos(nx)\cos(nx) + \ldots$$

But remember all the orange stuff above?

$$\int_0^\pi a_n \cos(nx)\cos(nx) = a_n \frac{\pi}{2}$$

But remember all the turqoise stuff above?

$$\frac{1}{2}(-1)^n \sin(z\pi)\left(\frac{1}{z-n} + \frac{1}{z+n}\right) = a_n \frac{\pi}{2}$$

And now, we have a formula for a_n where $n > 0$!

$$\frac{2}{\pi}\frac{1}{2}(-1)^n sin(z\pi)\left(\frac{1}{z-n} + \frac{1}{z+n}\right) = a_n = \frac{1}{\pi}(-1)^n sin(z\pi)\left(\frac{1}{z-n} + \frac{1}{z+n}\right)$$

And now we just put it back in for the original Fourier series

$$\begin{aligned} cos(zx) = &\frac{1}{z\pi} sin(z\pi)\, cos(0x) \\ &+ \frac{1}{\pi}(-1)^1 sin(z\pi)\left(\frac{1}{z-1} + \frac{1}{z+1}\right) cos(x) \\ &+ \frac{1}{\pi}(-1)^2 sin(z\pi)\left(\frac{1}{z-2} + \frac{1}{z+2}\right) cos(2x) + \ldots \\ &+ \frac{1}{\pi}(-1)^n sin(z\pi)\left(\frac{1}{z-n} + \frac{1}{z+n}\right) cos(nx) + \ldots \end{aligned}$$

and setting $x = 0$, we get

$$1 = \frac{1}{z\pi} sin(z\pi) + \frac{1}{\pi}(-1)^1 sin(z\pi)\left(\frac{1}{z-1} + \frac{1}{z+1}\right) + \frac{1}{\pi}(-1)^2 sin(z\pi)\left(\frac{1}{z-2} + \frac{1}{z+2}\right) + \ldots$$

and FINALLY, multiplying by π and dividing by $\sin(\pi z)$, we get

$$\frac{\pi}{\sin(\pi z)} = \frac{1}{z} + (-1)^1\left(\frac{1}{z-1} + \frac{1}{z+1}\right) + (-1)^2\left(\frac{1}{z-2} + \frac{1}{z+2}\right) + \ldots$$

and writing in summation notation, we get

$$\frac{\pi}{\sin(\pi z)} = \frac{1}{z} + \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z-n} + \frac{1}{z+n}\right)$$

AND WHAT DOES THE RIGHT PART OF THIS EQUAL? WE SHOWED THAT IT EQUALS $\Gamma(z)\Gamma(1-z)$!

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \quad \square$$

Chapter 2

2.1 Factorials

Reminders: The factorial is defined as $n! = n \cdot (n-1) \cdots 2 \cdot 1$, and the gamma function is defined as $\displaystyle\int_0^\infty t^{x-1} e^{-t}\, d = \frac{x!}{x}$

2.1.1 The Infinite Product

The factorial function can also be represented as an infinite product. To start, we must accept that for any integer m,

$$\lim_{n\to\infty} \frac{n!(n+1)^m}{(n+m)!} = 1$$

This makes sense because both the top and the bottom have a n^m term. It turns out that this also holds for any complex number z.

$$1 = \lim_{n\to\infty} \frac{n!(n+1)^z}{(n+z)!}$$

Multiplying both sides by $z!$

$$z! = \lim_{n\to\infty} n! \frac{z!}{(n+z)!}(n+1)^z$$

Simplifying, we get

$$z! = \lim_{n\to\infty} \frac{1 \cdots n}{(1+z)\cdots(n+z)}(n+1)^z$$

Condensing into product notation, we get

$$z! = \lim_{n\to\infty} \left(\prod_{i=1}^{n} \frac{i}{i+z} \right) (n+1)^z$$

We can express $(n+1)^z$ as $\prod_{i=1}^{\infty} \frac{(n+1)^z}{n^z}$ which becomes obvious when we write out the product $\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{(n+1)^z}{n^z}$ where everything cancels except for $\lim\limits_{n\to\infty}(n+1)^z$. Condensing the product further we get

$$z! = \prod_{n=1}^{\infty} \frac{n}{n+z} \frac{(n+1)^z}{n^z}$$

We will write this as the equivalent expression

$$z! = \prod_{n=1}^{\infty} \frac{1}{1+\frac{z}{n}} \left(1+\frac{1}{n}\right)^z = \prod_{n=1}^{\infty} \frac{\left(1+\frac{1}{n}\right)^z}{1+\frac{z}{n}}$$

Using this definition, the Gamma Function can be written as

$$\Gamma(z) = \frac{1}{z} \prod_{n=1}^{\infty} \frac{\left(1+\frac{1}{n}\right)^z}{1+\frac{z}{n}}$$

2.1.2 The Digamma Function

The digamma function ψ is defined as the derivative of the log gamma function; $\frac{d}{dx} ln(\Gamma(x))$.

$$\ln(\Gamma(z)) = \ln\left(\frac{1}{z} \prod_{n=1}^{\infty} \frac{(1+\frac{1}{n})^z}{1+\frac{z}{n}}\right)$$

Using log rules, we get

$$-\ln(z) + \sum_{n=1}^{\infty} z \ln\left(1+\frac{1}{n}\right) - \ln\left(1+\frac{z}{n}\right)$$

Now taking the derivative with respect to z

$$-\frac{1}{z} + \sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right) - \frac{\frac{1}{n}}{1+\frac{z}{n}}$$

And simplifying we get

$$\psi = -\frac{1}{z} + \sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right) - \frac{1}{n+z}$$

For our purposes, the derivative of digamma function is more useful; ψ'

$$\psi' = \frac{1}{z^2} + \sum_{n=1}^{\infty} \frac{1}{(n+z)^2} = \sum_{n=0}^{\infty} \frac{1}{(n+z)^2}$$

Now, if we find an approximation to the sum of reciprocal of squares, we can approximate the gamma function.

2.1.3 Approximating ψ'

To approximate ψ', we will use $\sum \frac{1}{n^2}$ just for ease of writing and notation. We can write $\sum \frac{1}{n^2}$ as a sum of telescopic series, helping us approximate it.

$$\begin{aligned}\sum \frac{1}{n^2} &= \sum \frac{1}{n(n+1)} + \sum \frac{1}{n^2(n+1)} \\ &= \sum \left(\frac{1}{n} - \frac{1}{n+1}\right) + \sum \frac{1}{n^2(n+1)}\end{aligned}$$

The next telescopic term is $\sum\left(\frac{1}{n^2}-\frac{1}{(n+1)^2}\right)=\sum\frac{2n+1}{n^2(n+1)^2}$. Now we can design the series for the leftover term from above: $\sum\frac{1}{n^2(n+1)}$. $\sum\frac{1}{n^2(n+1)}-\sum\frac{2n+1}{n^2(n+1)^2}=\sum\frac{n}{n^2(n+1)^2}$ doesn't work because there is an n left in the numerator, so we do $\sum\frac{1}{n^2(n+1)}-\frac{1}{2}\sum\frac{2n+1}{n^2(n+1)^2}=\frac{1}{2}\sum\frac{1}{n^2(n+1)^2}$

$$\sum\frac{1}{n^2(n+1)}=\frac{1}{2}\sum\frac{2n+1}{n^2(n+1)^2}+\frac{1}{2}\sum\frac{1}{n^2(n+1)^2}$$
$$\sum\frac{1}{n^2}=\sum\left(\frac{1}{n}-\frac{1}{n+1}\right)+\frac{1}{2}\sum\left(\frac{1}{n^2}-\frac{1}{(n+1)^2}\right)+\frac{1}{2}\sum\frac{1}{n^2(n+1)^2}$$

The next telescopic term is $\sum\left(\frac{1}{n^3}-\frac{1}{(n+1)^3}\right)=\sum\frac{3n^2+3n+1}{n^3(n+1)^3}$. Now we can design the series for the leftover term from above: $\frac{1}{2}\sum\frac{1}{n^2(n+1)^2}$. We will leave the $\frac{1}{2}$ out for now and add it back on later. $\sum\frac{1}{n^2(n+1)^2}-\frac{1}{3}\sum\frac{3n^2+3n+1}{n^3(n+1)^3}=-\frac{1}{3}\sum\frac{1}{n^3(n+1)^3}$. The $\frac{1}{3}$ was put there to cancel out the n^2 and n and just leave the constant term. After we bring back the one half, and simplify, we get

$$\sum\frac{1}{n^2}=\sum\left(\frac{1}{n}-\frac{1}{n+1}\right)+\frac{1}{2}\sum\left(\frac{1}{n^2}-\frac{1}{(n+1)^2}\right)+\frac{1}{6}\sum\left(\frac{1}{n^3}-\frac{1}{(n+1)^3}\right)-\frac{1}{6}\sum\frac{1}{n^3(n+1)^3}$$

The next telescopic term is $\sum\left(\frac{1}{n^4}-\frac{1}{(n+1)^4}\right)=\sum\frac{4n^3+6n^2+4n+1}{n^4(n+1)^4}$. However, $\sum\frac{1}{n^3(n+1)^3}-\sum\frac{4n^3+6n^2+4n+1}{n^4(n+1)^4}$ will always have an n^3 in the numerator, so we move on. From now on, T_a will signify $\sum\left(\frac{1}{n^a}-\frac{1}{(n+1)^a}\right)$ to save space and effort.

The next telescopic term is $T_5=\frac{5n^4+10n^3+10n^2+5n+1}{n^5(n+1)^5}$. Now we can design the series for the leftover term from above: $-\frac{1}{6}\sum\frac{1}{n^3(n+1)^3}$. We will leave the $-\frac{1}{6}$ out for now and add it back on later. $\sum\frac{1}{n^3(n+1)^3}-\frac{1}{5}\frac{5n^4+10n^3+10n^2+5n+1}{n^5(n+1)^5}=-\frac{1}{5}\sum\frac{5n^2+5n+1}{n^5(n+1)^5}$. We added the $\frac{1}{5}$ to cancel out the n^4 and n^3. After we bring back the $-\frac{1}{6}$, and simplify, we get

$$\sum\frac{1}{n^2}=T_1+\frac{1}{2}T_2+\frac{1}{6}T_3-\frac{1}{30}T_5+\frac{1}{30}\sum\frac{5n^2+5n+1}{n^5(n+1)^5}$$

Now that we have enough precision in our approximation for $\sum\frac{1}{n^2}$, we can clarify the bounds of the sum and evaluate T_a. We redefine $T_a=\sum_{n=z}^{\infty}\left(\frac{1}{n^a}-\frac{1}{(n+1)^a}\right)$, and using the folding properties of telescopic sums, we get that

$$\sum_{n=z}^{\infty}\frac{1}{n^2}\left(\text{aka}\sum_{n=0}^{\infty}\frac{1}{(n+z)^2}\right)=\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{6z^3}-\frac{1}{30z^5}+\dots$$

Which we can write in terms of ψ' as

$$\psi'=\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{6z^3}-\frac{1}{30z^5}+\dots$$

2.1.4 Integrating Back to Γ

We will use big-O notation from now on, which looks like this: $\psi'=\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{6z^3}+\mathcal{O}\left(\frac{1}{z^5}\right)$ which means that there are higher order terms starting with some constant times $\frac{1}{z^5}$. Back to integration.

Integrating with respect to z, we get

$$\psi\left(\text{aka}\frac{d}{dz}\ln(\Gamma(z))\right) = \ln(z) - \frac{1}{2z} - \frac{1}{12z^2} + \mathcal{O}\left(\frac{1}{z^4}\right) + c_1$$

Integrating again with respect to z

$$\ln(\Gamma(z)) = z\ln(z) - z - \frac{1}{2}\ln(z) + \frac{1}{12z} + \mathcal{O}\left(\frac{1}{z^3}\right) + c_1 z + c_2$$

Grouping terms together

$$\ln(\Gamma(z)) = \left(z - \frac{1}{2}\right)\ln(z) - z + \frac{1}{12z} + \mathcal{O}\left(\frac{1}{z^3}\right) + c_1 z + c_2$$

On the first page of this document, I wrote the defining characteristic of the gamma function: $\Gamma(z) = \frac{z!}{z}$ or in other words $\frac{\Gamma(z+1)}{\Gamma(z)} = z$. Now taking the logarithm of both sides, we get $\ln\Gamma(z+1) - \ln\Gamma(z) = \ln z$. Using the equation above, we get

$$\left((z+1) - \frac{1}{2}\right)\ln(z+1) - (z+1) + \frac{1}{12(z+1)} + \mathcal{O}\left(\frac{1}{z^3}\right) + c_1(z+1) + c_2$$
$$- \left(\left(z - \frac{1}{2}\right)\ln(z) - z + \frac{1}{12z} + \mathcal{O}\left(\frac{1}{z^3}\right) + c_1 z + c_2\right) = \ln z$$

And distributing the negatives and simplifying a bit, we get

$$\left(z + \frac{1}{2}\right)\ln(z+1) - z - 1 + \frac{1}{12(z+1)} + \mathcal{O}\left(\frac{1}{z^3}\right) + c_1 z + c_1 + c_2$$
$$- \left(z - \frac{1}{2}\right)\ln(z) + z - \frac{1}{12z} + \mathcal{O}\left(\frac{1}{z^3}\right) - c_1 z - c_2 = \ln z$$

And simplifying more,

$$\left(z + \frac{1}{2}\right)\ln(z+1) - \left(z - \frac{1}{2}\right)\ln(z) + \frac{1}{12(z+1)} - \frac{1}{12z} + \mathcal{O}\left(\frac{1}{z^3}\right) + c_1 - 1 = \ln z$$

Subtracting the $\ln z$ from the right side (being extremely careful of the minus signs everywhere of course), and grouping terms together

$$\left(z + \frac{1}{2}\right)(\ln(z+1) - \ln(z)) + \frac{1}{12(z+1)} - \frac{1}{12z} + \mathcal{O}\left(\frac{1}{z^3}\right) + c_1 - 1 = 0$$

Now as we take the limit as z goes to infinity, we get

$$\lim_{z\to\infty}\left(z + \frac{1}{2}\right)(\ln(z+1) - \ln(z)) + c_1 - 1 = 0$$

Using log properties,

$$\lim_{z\to\infty} \ln\left(1+\frac{1}{z}\right)^{z+\frac{1}{2}} + c_1 - 1 = 0$$

And as one of the major definitions of $e = \lim_{z\to\infty}\left(1+\frac{1}{z}\right)^z$, the limit evaluates to

$$\ln e + c_1 - 1 = 1 + c_1 - 1 = c_1 = 0$$

Plugging in c_1

$$\ln(\Gamma(z)) = \left(z - \frac{1}{2}\right)\ln(z) - z + \frac{1}{12z} + \mathcal{O}\left(\frac{1}{z^3}\right) + c_2$$

We still have the c_2 constant, which we will figure out using Legendre's Duplication Formula:

$$\Gamma(x)\Gamma\left(x+\frac{1}{2}\right) = \sqrt{\pi}2^{1-2x}\Gamma(2x)$$

2.2 Legendre's Duplication Formula

To prove the duplication formula, we first begin with our old friend, the gamma reflection formula.

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \tag{2.1}$$

Secondly, we establish an identity for a product of sines:

$$\prod_{k=1}^{n-1}\sin\left(\frac{k\pi}{n}\right) = \frac{n}{2^{n-1}} \tag{2.2}$$

The famous complex exponential formula gives us that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. If we use $-\theta$ instead of θ, we get $e^{-i\theta} = \cos(\theta) - i\sin(\theta)$. If we subtract the second from the first, we get $e^{i\theta} - e^{-i\theta} = 2i\sin(\theta) \to \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \sin(\theta)$ and plugging in $\theta = \frac{k\pi}{n}$, we get

$$\frac{1}{2i}(e^{i\frac{k\pi}{n}} - e^{-i\frac{k\pi}{n}}) = \sin(\frac{k\pi}{n})$$

Plugging into the product above

$$\prod_{k=1}^{n-1}\frac{1}{2i}(e^{i\frac{k\pi}{n}} - e^{-i\frac{k\pi}{n}})$$

Manipulating, we get

$$\frac{1}{2^{n-1}}\prod_{k=1}^{n-1}\frac{e^{i\frac{k\pi}{n}}}{i}(1 - e^{-2i\frac{k\pi}{n}})$$

We know from looking at roots of unity that

$$z^n - 1 = \prod_{k=1}^{n} (z - e^{2i\frac{k\pi}{n}})$$

where $e^{2i\frac{k\pi}{n}}$ are the roots of unity, just like $z^4 - 1 = (z-1)(z+1)(z-1)(z+i) = \prod_{k=1}^{4}(z - e^{2i\frac{k\pi}{4}})$ where $1, -1, i, -i$ are the fourth roots of unity. Also, note that $\prod_{k=1}^{n}(z - e^{2i\frac{k\pi}{n}}) = \prod_{k=1}^{n}(z - e^{-2i\frac{k\pi}{n}})$ due to the symmetry of the unit circle; one goes clockwise from zero, one goes counterclockwise. Rearranging the formula a bit, we get

$$z^n - 1 = \prod_{k=1}^{n-1} (z - e^{-2i\frac{k\pi}{n}}) \cdot (z - e^{-2i\pi}) \to z^n - 1 = \prod_{k=1}^{n-1} (z - e^{-2i\frac{k\pi}{n}}) \cdot (z - 1)$$

which we can write as

$$\frac{z^n - 1}{z - 1} = \prod_{k=1}^{n-1} (z - e^{-2i\frac{k\pi}{n}})$$

Taking the limit as $z \to 1$, using L'Hopital's rule, we get

$$n = \prod_{k=1}^{n-1} (1 - e^{-2i\frac{k\pi}{n}})$$

Plugging in above, we get

$$\frac{n}{2^{n-1}} \prod_{k=1}^{n-1} \frac{e^{i\frac{k\pi}{n}}}{i}$$

$\displaystyle\prod_{k=1}^{n-1} \frac{e^{i\frac{k\pi}{n}}}{i}$ evaluates to

$$\frac{\exp(\sum_{k=1}^{n-1} \frac{i\pi}{n}(1 + 2 + 3 + \ldots + (n-1)))}{i^{n-1}}$$

which is just

$$\frac{\exp(\frac{i\pi}{n}\frac{n(n-1)}{2})}{i^{n-1}} = \frac{\exp(i\pi)(\frac{n-1}{2})}{i^{n-1}} = \frac{(-1)^{\frac{n-1}{2}}}{i^{n-1}} = \frac{i^{n-1}}{i^{n-1}} = 1$$

And finally, going back to the beginning, we get

$$\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right) = \frac{n}{2^{n-1}}$$

Now that we have all our identities written out, we will begin. We define a function $f(x)$

$$f(x) = \prod_{k=0}^{n-1} \Gamma\left(x + \frac{k}{n}\right) \tag{2.3}$$

Multiplying by x and manipulating, we get

$$\begin{aligned}
x\,f(x) &= x\prod_{k=0}^{n-1}\Gamma\left(x+\frac{k}{n}\right)\\
&= x\,\Gamma(x)\prod_{k=1}^{n-1}\Gamma\left(x+\frac{k}{n}\right)\\
&= \Gamma(x+1)\prod_{k=0}^{n-2}\Gamma\left(x+\frac{k+1}{n}\right)\\
&= \Gamma\left(x+\frac{(n-1)+1}{n}\right)\prod_{k=0}^{n-2}\Gamma\left(x+\frac{k+1}{n}\right)\\
&= \prod_{k=0}^{n-1}\Gamma\left(x+\frac{k+1}{n}\right)\\
&= \prod_{k=0}^{n-1}\Gamma\left(\left(x+\frac{1}{n}\right)+\frac{k}{n}\right)\\
&= f\left(x+\frac{1}{n}\right)
\end{aligned}$$

Changing the variable $x=\frac{x}{n}$, we get

$$\frac{x}{n}f\left(\frac{x}{n}\right)=f\left(\frac{x+1}{n}\right) \tag{2.4}$$

Multiplying $n^{x+1}\frac{1}{nf\left(\frac{1}{n}\right)}$ on both sides

$$xn^x f\left(\frac{x}{n}\right)\frac{1}{nf\left(\frac{1}{n}\right)}=n^{x+1}f\left(\frac{x+1}{n}\right)\frac{1}{nf\left(\frac{1}{n}\right)}$$

And if we set $G(x)=n^x f\left(\frac{x}{n}\right)\frac{1}{nf\left(\frac{1}{n}\right)}$, we get that

$$xG(x)=G(x+1)$$

Which is the gamma function identity! This hints that $G(x)=\Gamma(x)$. A fully rigorous proof of $G(x)=\Gamma(x)$ can be acheived through the Bohr-Mollerup Theorem, a theorem I will not prove in this paper. We can set

$$\Gamma(x)=n^x f\left(\frac{x}{n}\right)\frac{1}{nf\left(\frac{1}{n}\right)}$$

And manipulating, we get

$$f\left(\frac{x}{n}\right)=nf\left(\frac{1}{n}\right)\frac{\Gamma(x)}{n^x} \tag{2.5}$$

To find $f\left(\frac{1}{n}\right)$, we first find $f^2\left(\frac{1}{n}\right)$.

$$f^2\left(\frac{1}{n}\right) = \prod_{k=1}^{n-1} \Gamma\left(\frac{1}{n}+\frac{k}{n}\right)\Gamma\left(\frac{1}{n}+\frac{k}{n}\right)$$

where $\prod_{k=1}^{n-1} \Gamma\left(\frac{k}{n}\right) = \prod_{k=0}^{n-1} \Gamma\left(\frac{1}{n}+\frac{k}{n}\right)$ because $\Gamma((\frac{n}{n}) = \Gamma((1-\frac{0}{n}) = 1$ and $\prod_{k=1}^{n-1} \Gamma\left(\frac{k}{n}\right) = \prod_{k=1}^{n-1} \Gamma\left(1-\frac{k}{n}\right)$ because of symmetry.

$$f^2\left(\frac{1}{n}\right) = \prod_{k=1}^{n-1} \Gamma\left(\frac{k}{n}\right)\Gamma\left(1-\frac{k}{n}\right)$$

Using the reflection formula, (1), we get

$$f^2\left(\frac{1}{n}\right) = \prod_{k=1}^{n-1} \frac{\pi}{\sin(k\pi/n)}$$

Simplifying and using the product of sines, (2), we get

$$f^2\left(\frac{1}{n}\right) = \frac{1}{n}2^{n-1}\pi^{n-1}$$

Square rooting and multiplying by the n in (5)

$$f\left(\frac{x}{n}\right) = \sqrt{n2^{n-1}\pi^{n-1}}\frac{\Gamma(x)}{n^x}$$

and finally variable change $x = nx$, we get Gauss's multiplication formula, or the generalized version of the duplication formula

$$f(x) = \prod_{k=0}^{n-1} \Gamma\left(x+\frac{k}{n}\right) = \sqrt{n2^{n-1}\pi^{n-1}}\frac{\Gamma(nx)}{n^{nx}} \tag{2.6}$$

Setting $n = 2$, we get the duplication formula.

$$\Gamma(x)\Gamma\left(x+\frac{1}{2}\right) = \sqrt{\pi}2^{1-2x}\Gamma(2x)$$

2.3 The Grand Finale

Taking the natural log of the duplication formula

$$\ln\left(\Gamma(x)\right) + \ln\left(\Gamma\left(x+\frac{1}{2}\right)\right) = \ln\sqrt{\pi} + \ln(2^{1-2x}) + \ln(\Gamma(2x))$$

Taking our approximation of $\ln \Gamma(x)$ from Section 1

$$\ln(\Gamma(z)) = \left(z - \frac{1}{2}\right)\ln(z) - z + \frac{1}{12z} + \mathcal{O}\left(\frac{1}{z^3}\right) + c_2$$

and pluggin it in the log duplication formula, we get

$$\begin{aligned}\left(z - \frac{1}{2}\right)\ln(z) - z + \frac{1}{12z} + \mathcal{O}\left(\frac{1}{z^3}\right) + c_2 \\ + z\ln(z + \frac{1}{2}) - z - \frac{1}{2} + \frac{1}{12z+6} + \mathcal{O}\left(\frac{1}{z^3}\right) + c_2 \\ = \ln\sqrt{\pi} + \ln(2^{1-2x}) + \left(2z - \frac{1}{2}\right)\ln(2z) - z + \frac{1}{24z} + \mathcal{O}\left(\frac{1}{z^3}\right) + c_2\end{aligned}$$

Distributing and using log rules, we get

$$\begin{aligned}z\ln(z) - \frac{1}{2}\ln(z) - z + \frac{1}{12z} + \mathcal{O}\left(\frac{1}{z^3}\right) + c_2 \\ + z\ln(z + \frac{1}{2}) - z - \frac{1}{2} + \frac{1}{12z+6} + \mathcal{O}\left(\frac{1}{z^3}\right) + c_2 \\ = \ln\sqrt{\pi} + (1-2z)\ln(2) + \left(2z - \frac{1}{2}\right)\ln(2) + 2z\ln(z) - \frac{1}{2}\ln(z) - 2z + \frac{1}{24z} + \mathcal{O}\left(\frac{1}{z^3}\right) + c_2\end{aligned}$$

Simplifying,

$$z\ln(z + \frac{1}{2}) - z\ln(z) - \frac{1}{2} + \mathcal{O}\left(\frac{1}{z}\right) + c_2 = \ln\sqrt{\pi} + \frac{1}{2}\ln(2)$$

Simplifying more,

$$\ln\left(\frac{z + \frac{1}{2}}{z}\right)^z - \frac{1}{2} + \mathcal{O}\left(\frac{1}{z}\right) + c_2 = \ln\sqrt{2\pi}$$

And taking the limit as $z \to \infty$

$$\ln\left(e^{\frac{1}{2}}\right) - \frac{1}{2} + 0 + c_2 = \ln\sqrt{2\pi} \to c_2 = \ln\sqrt{2\pi}$$

And putting everything back together, we have

$$\ln(\Gamma(z)) = \left(z - \frac{1}{2}\right)\ln(z) - z + \frac{1}{12z} + \mathcal{O}\left(\frac{1}{z^3}\right) + \ln\sqrt{2\pi}$$

And exponentiating both sides,

$$\Gamma(z) = z^{z-\frac{1}{2}} \cdot e^{-z} \cdot \exp\left(\frac{1}{12z} + \mathcal{O}\left(\frac{1}{z^3}\right)\right) \cdot e^{\ln\sqrt{2\pi}}$$

And simplifying, we get

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}} \left(\frac{z}{e}\right)^z \exp\left(\frac{1}{12z} + \mathcal{O}\left(\frac{1}{z^3}\right)\right)$$

AND BECAUSE $z! = z\Gamma(z)$, WE GET

$$z! = \sqrt{2\pi z} \left(\frac{z}{e}\right)^z \exp\left(\frac{1}{12z} + \mathcal{O}\left(\frac{1}{z^3}\right)\right) \approx \sqrt{2\pi z} \left(\frac{z}{e}\right)^z \quad \square$$

and we are done.

Chapter 3

Prime numbers are notoriously difficult beasts to study. They appear rather unpredictably, sometimes appearing in bursts (e.g. twin primes) or sometimes not at all (e.g. $1000!+1, 2, 3, 4, \ldots, 999, 1000$). Furthermore, they are purely discrete entities, so it's not immediately obvious how we use the mathematical tools we've developed (calculus/analysis, mainly), which are mostly based on continuous functions, to study them. However, there's a particularly nice result that's not too difficult to prove that will serve nicely as an introduction to analytic number theory – Bertrand's Postulate: there must be at least one prime between any positive integer n and $2n$.

3.1 Some Useful Functions & Identities

I will now introduce the star functions of the show. Just a note about notation: p is a prime number; $n, i, d, k \in \mathbb{Z}^+$; and $x \in \mathbb{R}^+$. You might see me drop the $1 \leq \ldots$ on the bounds of sums sometimes, but because we are only looking at positive integers in those bounds, the $1 \leq \ldots$ is implicitly there.

Let us define a function $\Lambda(n)$ (the Von-Mangoldt function) as

$$\Lambda(n) = \begin{cases} \ln(p) & \text{if } n = p^k \text{ (for some prime } p \text{ and positive integer } k) \\ 0 & \text{otherwise} \end{cases}$$

The first couple values of $\Lambda(n)$ are

$$0, \ln 2, \ln 3, \ln 2, \ln 5, 0, \ln 7, \ln 2, \ln 3, 0, \ln(11), 0, \ln(13), 0, 0, \ln 2, \ln(17)$$

In plain English, $\Lambda(n)$ finds pure powers of primes and returns the log of the prime. As we will see, the log allows us to multiply prime factors together by sum over values of $\Lambda(n)$ because of the property that $\ln a + \ln b = \ln(ab)$.

Our next function, $\vartheta(x)$ (the first Chebyshev function), is defined as

$$\vartheta(x) = \sum_{p \leq x} \ln(p)$$

The first couple values of $\vartheta(x)$ evaluated at the integers are

$$0, \ln 2, \ln 6, \ln 6, \ln(30), \ln(30), \ln(210), \ln(210), \ln(210), \ln(210), \ln(2310), \ln(2310)$$

In other words, $\vartheta(x) = \ln(\text{product of all the primes} \leq x)$. Also, $\vartheta(x)$ is an increasing function.

Our last function, $\psi(x)$ (the second Chebyshev function), is defined as

$$\psi(x) = \sum_{n=1}^{\lfloor x \rfloor} \Lambda(n) = \sum_{n=1}^{x} \Lambda(n) = \sum_{n \leq x} \Lambda(n) = \sum_{p^k \leq x} \ln(p) = \sum_{i \geq 1} \sum_{p^i \leq x} \ln(p)$$

The first couple values of $\psi(x)$ evaluated at the integers are

$$0, \ln 2, \ln 6, \ln(12), \ln(60), \ln(60), \ln(420), \ln(840), \ln(2920), \ln(2920), \ln(32120), \ln(32120)$$

The definition $\psi(x) = \sum_{i \geq 1} \sum_{p^i \leq x} \ln p$ in English means that $\psi(x) = \ln(\text{the product of the highest pure prime powers} \leq x \text{ for every prime})$. Another equivalent definition that you can verify for yourself is the following:

$$\psi(x) = \ln(\text{LCM}(\text{all the integers from } 1 \ldots x))$$

Also, note that

$$\ln n = \sum_{d|n} \Lambda(d)$$

where $\sum_{d|n}$ means that we sum over all $d \in \mathbb{Z}^+$ that divide n (all the divisors of n). The identity makes sense because this is essentially just looking through the divisors of n, finding the pure prime powers (the others go to 0), and multiplying the corresponding primes together. More explicitly, if 3^3 is the highest pure power of 3 divisor of n, then Λ makes note of $3^1, 3^2$ and 3^3, a total of 3 times, in essence finding the prime factorization of n.

Futhermore, you have probably noticed that $\vartheta(x)$ is pretty similar to $\psi(x)$ $\left(\sum_{p \leq x} \text{versus } \sum_{p^k \leq x}\right)$, and naturally there's a nice identity that relates them (the clever trick is to see that for $p, x, i > 0$, $p \leq x^{1/i} \iff p^i \leq x$):

$$\exp\left(\sum_{i \geq 1} \vartheta(x^{1/i})\right) = \prod_{i \geq 1} \exp\left(\vartheta(x^{1/i})\right) = \prod_{i \geq 1} \exp\left(\sum_{p \leq x^{1/i}} \ln(p)\right) = \prod_{i \geq 1} \prod_{p \leq x^{1/i}} p$$
$$= \prod_{i \geq 1} \prod_{p^i \leq x} p = \exp\left(\ln\left(\prod_{i \geq 1} \prod_{p^i \leq x} p\right)\right) = \exp\left(\sum_{i \geq 1} \sum_{p^i \leq x} \ln p\right) = e^{\psi(x)}$$

which means that

$$\sum_{i\geq 1} \vartheta(x^{1/i}) = \psi(x) \tag{3.1}$$

Our final identity, building upon all our prior definitions and identities, is as follows:

$$\begin{aligned}
\ln(\lfloor x \rfloor !) &= \ln\left(\prod_{n=1}^{\lfloor x \rfloor} n\right) = \sum_{n=1}^{\lfloor x \rfloor} \ln(n) = \sum_{n\leq x} \ln(n) = \sum_{n\leq x}\sum_{d|n} \Lambda(d) = \sum_{n\leq x}\sum_{dk=n} \Lambda(d) \\
&= \sum_{dk\leq x} \Lambda(d) \text{ (i.e. summing over all distinct pairs } (d,k) \text{ where } dk \leq x) = \sum_{k=1}^{x}\sum_{dk\leq x} \Lambda(d) \\
&= \sum_{k=1}^{x}\left(\sum_{d\leq x/k} \Lambda(d)\right) = \sum_{k=1}^{x} \psi(x/k) = \sum_{k\geq 1} \psi\left(\frac{x}{k}\right)
\end{aligned}$$

We can ignore the upper bound on k in the final equality because once k exceeds x, $\psi(x/k)$ becomes 0 (just a little simplification to remove unnecessary information). The identity in its final form is

$$\ln(\lfloor x \rfloor !) = \sum_{k\geq 1} \psi\left(\frac{x}{k}\right) \tag{3.2}$$

3.2 Inequalities Galore!

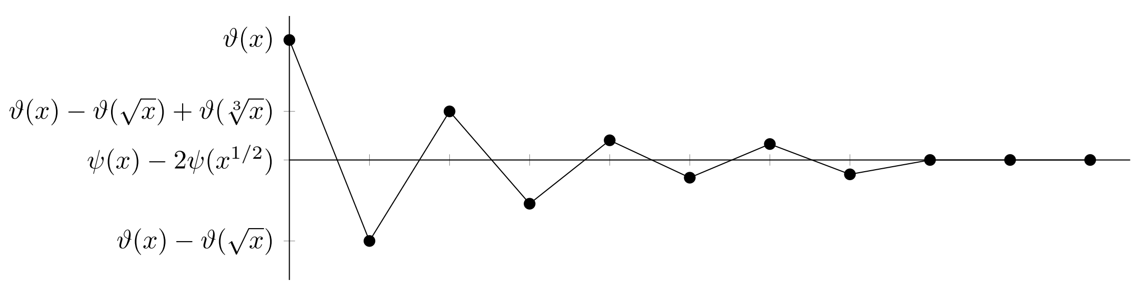
Now that I've set things up, our goal is to establish certain inequalities on our three star functions using functions we understand better. From equation (1) ($\psi(x) = \sum_{i\geq 1} \vartheta(x^{1/i})$), we get that

$$2\psi(x^{1/2}) = 2\sum_{i\geq 1} \vartheta(x^{1/2i})$$

Subtracting this from (1), all the even numbered terms turn negative as follows

$$\psi(x) - 2\psi(x^{1/2}) = \vartheta(x) - \vartheta(\sqrt{x}) + \vartheta(\sqrt[3]{x}) - \dots$$

As I stated above, $\vartheta(x)$ is increasing, so for $x > 1$ and $i < j$, $x^{1/i} > x^{1/j} \implies \vartheta(x^{1/i}) \geq \vartheta(x^{1/j})$. For $x \leq 1$, $\vartheta(x)$ and $\psi(x)$ are 0. Thus the sum is an alternating series with the magnitude of the terms decreasing. Visualizing this, we get something like this (it's not exact, but you get the idea):



From the figure (or just thinking about the fact that the series is alternating with the magnitude of the terms decreasing), it's clear that $\vartheta(x)$ is greater than $\psi(x) - 2\psi(x^{1/2})$. Furthermore, we know that

$$\vartheta(x) = \sum_{p^1 \le x} \ln(p) \le \sum_{i \ge 1} \sum_{p^i \le x} \ln(p) = \psi(x)$$

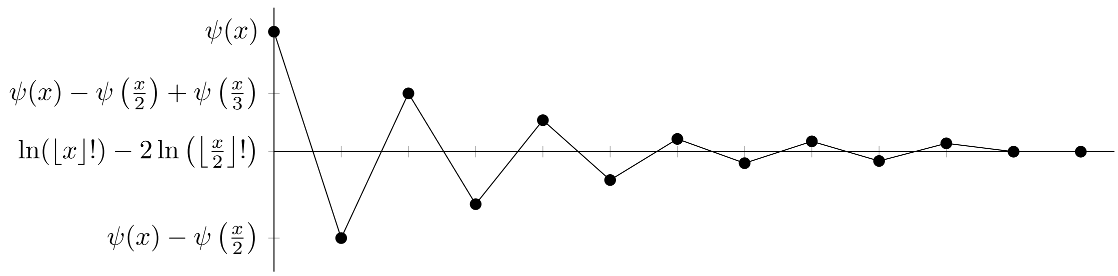
so we have our first bounds on $\vartheta(x)$:

$$\psi(x) - 2\psi(x^{1/2}) \le \vartheta(x) \le \psi(x) \tag{3.3}$$

Similarly, using equation (2), we can get that

$$\ln(\lfloor x \rfloor !) - 2\ln\left(\left\lfloor \frac{x}{2} \right\rfloor !\right) = \sum_{k \ge 1} \psi\left(\frac{x}{k}\right) - 2\sum_{k \ge 1} \psi\left(\frac{x}{2k}\right) = \psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right) - \dots$$

I'll show a not-exact plot for reference:



from which we can see clearly that

$$\psi(x) - \psi\left(\frac{x}{2}\right) \le \ln(\lfloor x \rfloor !) - 2\ln\left(\left\lfloor \frac{x}{2} \right\rfloor !\right) \le \psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right) \tag{3.4}$$

Now that we've established those two bounds, let's use our beloved Gamma function $\Gamma(x)$ to rework some of the inequalities involving $\lfloor x \rfloor !$ — keeping in mind that $n! = \Gamma(n+1) = n\Gamma(n)$.

For $x \ge 1$, $\Gamma(x)$ is increasing: $x \le y \implies \Gamma(x) \le \Gamma(y)$ (hence preserving inequalities), so

$$\begin{aligned} x - 1 \le \lfloor x \rfloor \le x &\implies x \le \lfloor x \rfloor + 1 \le x + 1 \\ &\implies \Gamma(x) \le \Gamma(\lfloor x \rfloor + 1) \le \Gamma(x+1) \\ &\implies \Gamma(x) \le \lfloor x \rfloor ! \le \Gamma(x+1) \end{aligned}$$

The second half of the inequality applied to $\frac{x}{2} - \frac{1}{2}$ yields that for $x \ge 2$,

$$\Gamma\left(\frac{1}{2}x - \frac{1}{2}\right) \le \left\lfloor \frac{x}{2} - \frac{1}{2} \right\rfloor ! \le \Gamma\left(\frac{1}{2}x + \frac{1}{2}\right)$$

Armed with these inequalities

$$\ln\Gamma(x+1) - 2\ln\Gamma\left(\frac{1}{2}x+\frac{1}{2}\right) \geq \ln(\lfloor x\rfloor!) - 2\ln\left(\left\lfloor\frac{x}{2}\right\rfloor!\right) \geq \ln\Gamma(x) - 2\ln\Gamma\left(\frac{1}{2}x+\frac{1}{2}\right)$$

as pictured below:

3.3 Stirling's Approximation

Now that we have our inequalities all listed out, we can use Stirling's Aprroximation. Recall that

$$n\Gamma(n) = n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Thus

$$\Gamma(n) = \sqrt{\frac{2\pi}{n}}\left(\frac{n}{e}\right)^n$$

Using it in our first inequality,

$$\frac{1}{2}\ln\left(\frac{2\pi}{n+1}\right) + (n+1)\ln\left(\frac{n+1}{e}\right) - \ln\left(\frac{4\pi}{n+1}\right) - (n+1)\ln\left(\frac{n+1}{2e}\right) \geq \ln(\lfloor n\rfloor!) - 2\ln\left(\left\lfloor\frac{n}{2}\right\rfloor!\right)$$

which is just

$$\frac{1}{2}\ln\left(\frac{n+1}{8\pi}\right) + (n+1)\ln 2 \geq \ln(\lfloor n\rfloor!) - 2\ln\left(\left\lfloor\frac{n}{2}\right\rfloor!\right)$$

Using it in our second inequality,

$$\frac{1}{2}\ln\left(\frac{2\pi}{n}\right) + n\ln\left(\frac{n}{e}\right) - \ln\left(\frac{4\pi}{n+1}\right) - (n+1)\ln\left(\frac{n+1}{2e}\right) \leq \ln(\lfloor n\rfloor!) - 2\ln\left(\left\lfloor\frac{n}{2}\right\rfloor!\right)$$

which is just

$$\frac{1}{2}\ln\left(\frac{(n+1)^2}{8\pi n}\right) + (n+1)\ln\left(\frac{2n}{n+1}\right) - \ln\left(\frac{n}{e}\right) \leq \ln(\lfloor n\rfloor!) - 2\ln\left(\left\lfloor\frac{n}{2}\right\rfloor!\right)$$

These two new bounds are easier to work with as they are just logarithmic functions compared to Gamma functions; a bit of luck (or some graphing and deriving to ensure the difference the linear and logarithmic bounds is increasing and thus will always enclose $\ln(\lfloor n\rfloor!) - 2\ln\left(\left\lfloor\frac{n}{2}\right\rfloor!\right)$) will reveal that for big enough x,

$$\frac{2}{3}x < \ln(\lfloor n\rfloor!) - 2\ln\left(\left\lfloor\frac{n}{2}\right\rfloor!\right) < \frac{3}{4}x \qquad (3.5)$$

3.4 Finale

From (5),

$$\psi(x) - \psi\left(\frac{x}{2}\right) \leq \ln(\lfloor x \rfloor!) - 2\ln\left(\left\lfloor \frac{x}{2} \right\rfloor!\right) \leq \psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right)$$

and from (6),

$$\frac{2}{3}x < \ln(\lfloor n \rfloor!) - 2\ln\left(\left\lfloor \frac{n}{2} \right\rfloor!\right) < \frac{3}{4}x$$

which means that

$$\psi(x) - \psi\left(\frac{x}{2}\right) < \frac{3}{4}x, \text{ and for } x \text{ greater than say 200, } \frac{2}{3}x < \psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right) \tag{3.6}$$

Summing the first inquality over x values that half each time yields

$$\psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{2}\right) - \psi\left(\frac{x}{4}\right) + \psi\left(\frac{x}{4}\right) - \psi\left(\frac{x}{8}\right) + \ldots < \frac{3}{4}x + \frac{3}{8}x + \frac{3}{16} + \ldots$$

which means

$$\psi(x) < \frac{3}{2}x \text{ for } x > 0 \tag{3.7}$$

From (4),

$$\psi(x) \geq \vartheta(x) \geq \psi(x) - 2\psi(x^{1/2})$$

Thus, we have

$$\begin{aligned} \psi(x) &\leq \vartheta(x) + 2\psi(\sqrt{x}) \\ -\psi\left(\frac{x}{2}\right) &\leq -\vartheta\left(\frac{x}{2}\right) \\ \psi\left(\frac{x}{3}\right) &= \psi\left(\frac{x}{3}\right) \end{aligned}$$

Adding everything, we get that

$$\psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right) \leq \vartheta(x) + 2\psi(\sqrt{x}) - \vartheta\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right)$$

Using (8), we can simplify the inequality to be

$$\psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right) < \vartheta(x) - \vartheta\left(\frac{x}{2}\right) + \frac{x}{2} + 3\sqrt{x}$$

For large x, we can use the second inequality of (7)

$$\frac{2}{3}x < \vartheta(x) - \vartheta\left(\frac{x}{2}\right) + \frac{x}{2} + 3\sqrt{x}$$

which yields

$$\vartheta(x) - \vartheta(x/2) > \frac{1}{6}x - 3\sqrt{x} \tag{3.8}$$

And from basic algebra and/or graphing

$$\frac{1}{6}x - 3\sqrt{x} \geq 0, \text{ if } x \geq 324, \text{ which is in fact large enough}$$

we know that for $x = 2n$ $n \geq 162$,

$$\vartheta(2n) - \vartheta(n) > 0 \quad \square$$

WHICH MEANS THERE MUST BE A PRIME IN BETWEEN n AND $2n$ for $n \geq 162$!!! We can very easily verify that for $n < 162$, there is indeed a prime between n and $2n$, thus proving once and for all Bertrand's Postulate!