The Γ Function, 0.5 edition.

Daniel Rui

1 The Gamma Function

Reminders: We can prove (using integraton by parts) that: $\Gamma(x+1) = x\Gamma(x)$ (It's not even very difficult to prove!)

This is one of the properties of the Gamma Function that tells us it is related to the factorial function.

The gamma function is defined to be:

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\, dt = (x-1)!$$

The whole purpose of the Gamma Function was to continuously define the factorial function for all numbers. Technically, we should be able to find $\frac{1}{2}!$, also known as $\Gamma\left(\frac{3}{2}\right)$, but as you may have guessed, this:

$$\int_0^\infty \sqrt{t} e^{-t}\, dt$$

is not a very easy integral to solve! And guess what we do in this paper? We figure it out.

2 The Reflection Formula

Let's begin with another introduction. This is the gamma reflection formula:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{sin(\pi z)}$$

With the gamma reflection formula, solving for $\frac{1}{2}!$ is really easy. Set $z = \frac{1}{2}$ to get:

$$\Gamma\left(\frac{1}{2}\right)\Gamma\left(1-\frac{1}{2}\right) = \frac{\pi}{sin(\frac{\pi}{2})}$$

so $\Gamma(\frac{1}{2})^2 = \pi$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, and because $\Gamma(x+1) = x\Gamma(x)$, $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$, so

$$\frac{1}{2}! = \frac{\sqrt{\pi}}{2} \quad \square$$

Now that we have established that we can find one half factorial from the reflection formula, we set out to prove it.

2.1 Integration

First things first, we plug in numbers into the definition of the gamma function, but we use different variables, in this case, t and s to allow combining the integrals together, which works because the s integral is a 'constant' according the the t integral, and constants can be pulled inside, allowing us to simplify two integrals into just one integral with two variables

$$
\begin{aligned}
\Gamma(z)\Gamma(1-z) &= \int_0^\infty s^{z-1}e^{-s}\,ds \int_0^\infty t^{(1-z)-1}e^{-t}\,dt \\
&= \int_0^\infty \left(\int_0^\infty s^{z-1}e^{-s}\,ds\right) t^{-z}e^{-t}\,dt \\
&= \int_0^\infty \int_0^\infty s^{z-1}e^{-s}t^{-z}e^{-t}\,ds\,dt \\
&= \int_0^\infty \int_0^\infty \frac{s^{z-1}}{t^z}e^{-(s+t)}\,ds\,dt \\
&= \int_0^\infty \int_0^\infty \frac{s^{z-1}}{t^{z-1}}\frac{1}{t}e^{-(s+t)}\,ds\,dt
\end{aligned}
$$

We do a variable change $u = s + t$ and $v = \frac{s}{t}$. Notice that $1 + v = \frac{u}{t}$, so $\frac{1+v}{u} = \frac{1}{t}$.

$$
\int_0^\infty \int_0^\infty \frac{v^{z-1}}{u}(1+v)e^{-u}\,ds\,dt
$$

In order to go from $ds\,dt$ to $du\,dv$ however,we have to do something fancier: the Jacobian

$$
du\,dv = \det\begin{bmatrix} \frac{\partial u}{\partial t} & \frac{\partial v}{\partial t} \\ \frac{\partial u}{\partial s} & \frac{\partial v}{\partial s} \end{bmatrix} ds\,dt
$$

Where det means determinant and ∂ is just the partial derivative. For more info on Jacobians, visit https://www.quora.com/What-is-an-intuitive-explanation-of-Jacobians-and-a-change-of-basis. Filling the matrix out, we get

$$
du\,dv = \det\begin{bmatrix} 1 & -\frac{s}{t^2} \\ 1 & \frac{1}{t} \end{bmatrix} ds\,dt
$$

Calculating, we get

$$\begin{aligned} du\,dv &= \frac{1}{t} + \frac{s}{t^2} ds\,dt \\ &= \frac{t+s}{t^2} ds\,dt \\ &= u\frac{(1+v)^2}{u^2} ds\,dt \\ &= \frac{(1+v)^2}{u} ds\,dt \end{aligned}$$

And rearranging to be more useful,

$$\frac{1}{1+v} du\,dv = \frac{1+v}{u} ds\,dt$$

and substituting back, we get

$$\int_0^\infty \int_0^\infty \frac{v^{z-1}}{1+v} e^{-u}\,du\,dv$$

Now that we have everything in u and v, we can pull the the integrals apart, the same way we originally combined them

$$\int_0^\infty \frac{v^{z-1}}{1+v}\,dv \int_0^\infty e^{-u}\,du$$

The u integral tells us that it goes to 1 (verify on your own), so all we have left is our v integral:

$$\int_0^\infty \frac{v^{z-1}}{1+v}\,dv$$

We can split it up into two integrals:

$$\int_0^1 \frac{v^{z-1}}{1+v}\,dv + \int_1^\infty \frac{v^{z-1}}{1+v}\,dv$$

And then if we do b-substitution on the second integral where $v = \frac{1}{b}$ (so $dv = \frac{-1}{b^2}db$), we get:

$$\int_1^0 \frac{b^{1-z}}{\frac{b+1}{b}} \frac{-1}{b^2}\,db = \int_1^0 \frac{b^{2-z}}{b+1} \frac{-1}{b^2}\,db = \int_0^1 \frac{b^{-z}}{b+1}\,db$$

Now, we can rename b to v (yes, we can do that), to get:

$$\int_0^1 \frac{v^{z-1}}{1+v}\,dv + \int_0^1 \frac{v^{-z}}{v+1}\,dv = \int_0^1 \frac{v^{z-1}+v^{-z}}{1+v}\,dv$$

But $\frac{1}{1+v}$ can be represented as an infinite geometric series: $\sum\limits_{n=0}^{\infty}(-v)^n$, and replacing the fraction with the sum, we get:

$$\int_0^1 v^{z-1}+v^{-z}\sum_{n=0}^{\infty}(-v)^n\,dv$$

In the sum's eyes, $v^{z-1}+v^{-z}$ is a constant because it doesn't have any n's in it, so we can pull it inside:

$$\int_0^1 \sum_{n=0}^{\infty}(v^{z-1}+v^{-z})(v)^n(-1)^n\,dv$$

But, the integral of the sum is the sum of the integrals!

$$\sum_{n=0}^{\infty}\int_0^1 (v^{z-1}+v^{-z})(v)^n(-1)^n\,dv$$

In the integral's eyes, the $(-1)^n$ is a constant because it doesn't have any v's:

$$\sum_{n=0}^{\infty}(-1)^n\int_0^1 (v^{z-1}+v^{-z})\,v^n\,dv$$

And distributing the $v^n\ldots$

$$\sum_{n=0}^{\infty}(-1)^n\int_0^1 (v^{n+z-1}+v^{-z+n})\,dv$$

then integrating (power rule)

$$\sum_{n=0}^{\infty}(-1)^n\left(\frac{v^{n+z}}{n+z}+\frac{v^{-z+n+1}}{-z+n+1}\right)\Bigg|_{v=0}^{v=1}$$

and evaluating . . .

$$\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{n+z}+\frac{1}{n+1-z}\right)$$

and then writing it out and shrinking it again (verify on your own)

$$\frac{1}{z}+\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n+z}+\frac{1}{n-z}\right)$$

We now take a step back, and look at another way to approach the problem: integrals of cosines.

2.2 Fourier Series

Fourier Series are all about representing any periodic function as $\sum_{n=0}^{\infty} a_n\, cos(nx)$. Let's start with $cos(zx)$, where $z \in \Re$ and z is not necessarily an integer.

2.2.1 Trig Integration

Note: all following integrals without a dx should have a dx.

Trig flashback: recall that

$$cos^2(x) + sin^2(x) = 1$$

and

$$cos(\alpha+\beta) = cos(\alpha)cos(\beta) - sin(\alpha)sin(\beta)$$

Setting $\alpha = \beta = x$, we get

$$cos(2x) = cos^2(x) - sin^2(x)$$

Adding the first equation and the one right above this sentence, we get

$$2cos^2(x) = 1 + cos(2x)$$

And simplifying, we get

$$cos^2(x) = \frac{cos(2x) + 1}{2}$$

Keep this in mind as we keep going.

Now the Fourier series,

$$\int_0^\pi cos(zx)cos(nx)\,dx$$

Integration by parts (do this on your own) gives

$$\left.\frac{\frac{1}{n}cos(zx)sin(nx) - \frac{z}{n^2}cos(nx)sin(zx)}{1 - \frac{z^2}{n^2}}\right|_{x=0}^{x=\pi}$$

If n and z are integers and $n \neq z$, then everything is zero.

If $n = z$ and n and z are integers, we get

$$\int_0^\pi cos^2(nx)\,dx = \frac{1}{2}\int_0^\pi cos(2nx) + 1\,dx$$

$$= \frac{1}{2}\left(\frac{1}{2}sin(2nx) + x\right)\Big|_0^\pi = \frac{\pi}{2}$$

If n is an integer and z not necessarily an integer, $sin(n\pi)$ will always be 0, and $cos(n\pi)$ will go back and forth between -1 and 1, so we can replace it with $(-1)^n$

$$-(-1)^n\frac{zsin(z\pi)}{n^2 - z^2}$$

Splitting $\frac{1}{n^2-z^2}$ into two fractions, we get $\frac{1}{2z(n-z)} - \frac{1}{2z(z+n)}$, and because $n - z = -(z - n)$, we can simplify

$$-(-1)^n z\,sin(z\pi)\left(-\frac{1}{2z(z-n)} - \frac{1}{2z(z+n)}\right) = \frac{1}{2}(-1)^n sin(z\pi)\left(\frac{1}{z-n} + \frac{1}{z+n}\right)$$

To find the Fourier Series of $cos(zx)$ we need to find a_n's such that

$$cos(zx) := a_0\, cos(0x) + a_1\, cos(x) + a_2\, cos(2x) + \ldots + a_n\, cos(nx) + \ldots$$

First, let's find a_0. Take $cos(zx) := a_0\, cos(0x) + a_1\, cos(x) + a_2\, cos(2x) + \ldots + a_n\, cos(nx) + \ldots$, and integrate it from 0 to π.

$$\int_0^{\pi} cos(zx) = \int_0^{\pi} a_0\, cos(0x) + \int_0^{\pi} a_1\, cos(x) + \ldots + \int_0^{\pi} a_n\, cos(nx) + \ldots$$

but what is $\int_0^{\pi} cos(nx)$ where n is an integer but not 0? It's just 0! (Think graphically; all the peaks cancel out the troughs)

$$\int_0^{\pi} cos(zx) = \int_0^{\pi} a_0\, dx \to \frac{1}{z} sin(zx)\Big|_0^{\pi} = \pi a_0$$

Evaluating, we get

$$a_0 = \frac{1}{z\pi} sin(z\pi)$$

Next, let's find a_n. First multiply everything by $cos(nx)$

$$cos(zx)cos(nx) = a_0\, cos(nx) + a_1\, cos(x)cos(nx) + \ldots + a_n\, cos(nx)cos(nx) + \ldots$$

And then integrate from 0 to π

$$\int_0^{\pi} cos(zx)cos(nx) = \int_0^{\pi} a_0\, cos(nx) + \int_0^{\pi} a_1\, cos(x)cos(nx) + \ldots + \int_0^{\pi} a_n\, cos(nx)cos(nx) + \ldots$$

But remember the red stuff above? All the $cos(\text{integer}x)cos(nx)$ are all zero!

$$\int_0^\pi cos(zx)cos(nx) = 0 + 0 + \ldots + \int_0^\pi a_n\, cos(nx)cos(nx) + \ldots$$

But remember all the orange stuff above?

$$\int_0^\pi cos(zx)cos(nx) = a_n \frac{\pi}{2}$$

But remember all the turqoise stuff above?

$$\frac{1}{2}(-1)^n sin(z\pi)\left(\frac{1}{z-n} + \frac{1}{z+n}\right) = a_n \frac{\pi}{2}$$

And now, we have a formula for a_n where $n > 0$!

$$\frac{2}{\pi}\frac{1}{2}(-1)^n sin(z\pi)\left(\frac{1}{z-n} + \frac{1}{z+n}\right) = a_n = \frac{1}{\pi}(-1)^n sin(z\pi)\left(\frac{1}{z-n} + \frac{1}{z+n}\right)$$

And now we just put it back in for the original Fourier series

$$\begin{aligned} cos(zx) = &\frac{1}{z\pi} sin(z\pi)\, cos(0x) \\ &+ \frac{1}{\pi}(-1)^1 sin(z\pi)\left(\frac{1}{z-1} + \frac{1}{z+1}\right) cos(x) \\ &+ \frac{1}{\pi}(-1)^2 sin(z\pi)\left(\frac{1}{z-2} + \frac{1}{z+2}\right) cos(2x) + \ldots \\ &+ \frac{1}{\pi}(-1)^n sin(z\pi)\left(\frac{1}{z-n} + \frac{1}{z+n}\right) cos(nx) + \ldots \end{aligned}$$

and setting $x = 0$, we get

$$1 = \frac{1}{z\pi} sin(z\pi) + \frac{1}{\pi}(-1)^1 sin(z\pi)\left(\frac{1}{z-1} + \frac{1}{z+1}\right) + \frac{1}{\pi}(-1)^2 sin(z\pi)\left(\frac{1}{z-2} + \frac{1}{z+2}\right) + \ldots$$

and FINALLY, multiplying by π and dividing by $sin(\pi z)$, we get

$$\frac{\pi}{sin(\pi z)} = \frac{1}{z} + (-1)^1\left(\frac{1}{z-1} + \frac{1}{z+1}\right) + (-1)^2\left(\frac{1}{z-2} + \frac{1}{z+2}\right) + \dots$$

and writing in summation notation, we get

$$\frac{\pi}{sin(\pi z)} = \frac{1}{z} + \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z-n} + \frac{1}{z+n}\right)$$

AND WHAT DOES THE RIGHT PART OF THIS EQUAL? WE SHOWED THAT IT EQUALS $\Gamma(z)\Gamma(1-z)$!

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{sin(\pi z)} \quad \square$$

and at last, we are done.