CTY LIN: Matrix Exponentials and the Fibonacci Sequence

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1 Matrix Exponentials

Reminders: Based on the Taylor Series for e^x, and replacing x with a matrix, A, we get a way to calculate e to some matrix:

$$e^A = \sum_{m=0}^{\infty} \frac{A^m}{m!} = I_n + A + \frac{A^2}{2!} + \dots$$

1.1 Compute e^{0_n}

Notice that $(0_n)^n = 0_n$.

$$e^{0_n} = I_n + 0_n + 0_n + \dots = I_n$$

1.2 Compute e^{I_n}

Notice that $(I_n)^n = I_n$, since $I_n A = A$.

$$e^{0_n} = I_n + I_n + \frac{I_n}{2!} + \dots = I_n \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots\right) = e(I_n)$$

1.3 Show that sometimes $e^A e^B \neq e^{A+B}$

We are given:

$$A = \begin{bmatrix} 0 & 1 & 7 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Also, because $A^3 = 0_3$ and likewise for B, A^n for $n \geq 3$ will also be zero, because no matter what you multiply with the zero matrix, it will also be the zero matrix (likewise for B).

e^A is as follows:

$$\begin{aligned} e^A &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 7 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0+0+0 & 0+0+0 & 0+2+0 \\ 0+0+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+0 \end{bmatrix} + 0_n + \dots \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 7 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 0_n + \dots \\ &= \begin{bmatrix} 1 & 1 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

e^B:

$$\begin{aligned} e^B &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0+0+0 & 0+0+0 & 0+3+0 \\ 0+0+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+0 \end{bmatrix} + 0_n + \dots \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1.5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 0_n + \dots \\ &= \begin{bmatrix} 1 & 1 & 0.5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$e^A e^B$:

$$\begin{aligned} e^A e^B &= \begin{bmatrix} 1 & 1 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0.5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1+0+0 & 1+1+0 & 0.5+3+8 \\ 0+0+0 & 0+1+0 & 0+3+2 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & 11.5 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

OK. For part two, we first need to calculate $A+B$.

$$\begin{bmatrix} 0 & 1 & 7 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

Now for the exponential:

$$\begin{aligned} e^{A+B} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0+0+0 & 0+0+0 & 0+10+0 \\ 0+0+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+0 \end{bmatrix} + \frac{(A+B)^3}{3!} + \dots \\ &= \begin{bmatrix} 1 & 2 & 6 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{3!}\begin{bmatrix} 0+0+5\cdot 0 & 0\cdot 2+0+5\cdot 0 & 0\cdot 6+0\cdot 5+5\cdot 0 \\ 0+0+0 & 0\cdot 2+0+0 & 0\cdot 6+0\cdot 5+0 \\ 0+0+0 & 0\cdot 2+0+0 & 0\cdot 6+0\cdot 5+0 \end{bmatrix} + \dots \\ &= \begin{bmatrix} 1 & 2 & 11 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix} + 0_n + \dots \\ &= \begin{bmatrix} 1 & 2 & 11 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

And because $\begin{bmatrix} 1 & 2 & 11.5 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 & 11 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix}$, $e^A e^B$ is not necessarily equal to e^{A+B}.

1.4 Show that $e^D = \begin{bmatrix} e^{d_1} & 0 & \dots & 0 \\ 0 & e^{d_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{d_n} \end{bmatrix}$

Matrix multiplication is defined as $(AB)_{ij} = \sum_{k=1}^{m} A_{ik}B_{kj}$. In the case of D, the diagonal matrix, every term except the terms of the form A_{kk} are zero, so the only terms in D^n will be the terms of the form A_{kk}, and those terms are going to be $(D_{kk})^n$. So when we raise the diagonal matrix D to any integer power, we are going to get

$$D^n = \begin{bmatrix} {d_1}^n & 0 & \dots & 0 \\ 0 & {d_2}^n & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & {d_m}^n \end{bmatrix}$$

Now, we are ready to tackle this problem.

$$e^D = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} + \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{bmatrix} + \frac{1}{2} \begin{bmatrix} {d_1}^2 & 0 & \dots & 0 \\ 0 & {d_2}^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & {d_n}^2 \end{bmatrix} + \dots$$

$$(e^D)_{11} = 1 + d_1 + \frac{{d_1}^2}{2!} + \frac{{d_1}^3}{3!} + \dots = e^{d_1}$$

The same reasoning can be applied to prove that $(e^D)_{22} = e^{d_2}$, $(e^D)_{33} = e^{d_3}$, and so on, up to $(e^D)_{nn} = e^{d_n}$. Thus,

$$e^D = \begin{bmatrix} e^{d_1} & 0 & \dots & 0 \\ 0 & e^{d_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{d_n} \end{bmatrix}$$

1.5 Show that $e^A = Pe^DP^{-1}$

The key insight here is that $A^n = PD^nP^{-1}$. When we raise A^n, we put lots of PDP^{-1} back to back, causing the $P^{-1}P$ to annihilate, thus leaving the P and P^{-1} on the ends and exactly n matrices D stuck between them.

$$\begin{aligned} e^A &= I_n + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots \\ &= I_n + PDP^{-1} + \frac{PD^2P^{-1}}{2!} + \frac{PD^3P^{-1}}{3!} + \dots \\ &= P\left(I_n + D + \frac{D^2}{2!} + \frac{D^3}{3!} + \dots\right)P^{-1} \\ &= Pe^DP^{-1} \end{aligned} \qquad \square$$

2 The Fibonacci Sequence

Reminders: The Fibonacci Sequence recursion $F_n = F_{n-1} + F_{n-2}$ can be represented using matrices, where $\vec{x}_n = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$:

$$\vec{x}_n = A^n \vec{x}_0$$

2.1 Finding the eigenvalues

The eigenvalues of A:

$$\det \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = 0$$

$$(1-\lambda)(-\lambda) - 1 = 0$$

$$\lambda^2 - \lambda - 1 = 0$$

Via the quadratic formula, we get that $\lambda = \frac{1 \pm \sqrt{5}}{2}$, or $\lambda = \varphi$, $\lambda = \bar{\varphi}$.

2.2 Finding the eigenvectors

The first eigenvector of A:

$$\begin{bmatrix} 1-\varphi & 1 \\ 1 & -\varphi \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 - x_2\varphi = 0$$

$$x_1 = x_2\varphi$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2\varphi \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \varphi \\ 1 \end{bmatrix}$$

$\begin{bmatrix} \varphi \\ 1 \end{bmatrix}$ is a eigenvector. The second is:

$$\begin{bmatrix} 1-\bar{\varphi} & 1 \\ 1 & -\bar{\varphi} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 - x_2\varphi = 0$$

$$x_1 = x_2\bar{\varphi}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2\bar{\varphi} \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \bar{\varphi} \\ 1 \end{bmatrix}$$

And $\begin{bmatrix} \bar{\varphi} \\ 1 \end{bmatrix}$ is the other eigenvector.

2.3 Diagonalize A

We know that $A = PDP^{-1}$ is true when P is a matrix of the eigenvectors of A and D is a matrix with the eigenvalues on the diagonal. We can verify that A is in fact PDP^{-1}.

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \varphi & \bar{\varphi} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \varphi & 0 \\ 0 & \bar{\varphi} \end{bmatrix} \begin{bmatrix} \varphi & \bar{\varphi} \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \varphi & \bar{\varphi} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \varphi & 0 \\ 0 & \bar{\varphi} \end{bmatrix} \left(\frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -\bar{\varphi} \\ -1 & \varphi \end{bmatrix} \right)$$

2.4 Fibonacci Formula

Now that we have an efficient way of raising A^n, we can easily find a formula for the Fibonacci Sequence.

$$\begin{aligned} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} &= A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} \varphi & \bar{\varphi} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \varphi & 0 \\ 0 & \bar{\varphi} \end{bmatrix}^n \begin{bmatrix} 1 & -\bar{\varphi} \\ -1 & \varphi \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned}$$

Now, we just multiply, right to left.

$$\begin{aligned}
\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} &= \frac{1}{\sqrt{5}} \begin{bmatrix} \varphi & \bar{\varphi} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \varphi^n & 0 \\ 0 & \bar{\varphi}^n \end{bmatrix} \begin{bmatrix} 1 & -\bar{\varphi} \\ -1 & \varphi \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{1}{\sqrt{5}} \begin{bmatrix} \varphi & \bar{\varphi} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \varphi^n & 0 \\ 0 & \bar{\varphi}^n \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
&= \frac{1}{\sqrt{5}} \begin{bmatrix} \varphi & \bar{\varphi} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \varphi^n \\ -\bar{\varphi}^n \end{bmatrix} \\
&= \frac{1}{\sqrt{5}} \begin{bmatrix} \varphi^{n+1} - \bar{\varphi}^{n+1} \\ \varphi^n - \bar{\varphi}^n \end{bmatrix}
\end{aligned}$$

Comparing terms of each vector, we can conclude that $F_n = \frac{1}{\sqrt{5}}(\varphi^n - \bar{\varphi}^n) \quad \square$.

2.5 F_{45}

Using Wolfram Alpha, we get that

$$\frac{1}{\sqrt{5}}(\varphi^{45} - \bar{\varphi}^{45}) = 1,134,903,170$$