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1 Problem G1

To find p_2, we find out which values for the second dice D_2 can fulfill the requirement that the product must be less than six given the value of D_1.

D_1	Probability of D_1	D_2	Probability of D_2
1	1/6	1,2,3,4,5,6	6/6
2	1/6	1,2,3	3/6
3	1/6	1,2	2/6
4	1/6	1	1/6
5	1/6	1	1/6
6	1/6	1	1/6

Multiplying the probabilities and adding, we get

$$\frac{6}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} = \frac{14}{36} = \frac{7}{18}$$

Now for p_n. Instead of two dice, we now have n dice. We split this up into six cases; one for each of the possible values of D_1.

D_1 is 1 with probability 1/6. In what scenario will the entire product be ≤ 6? Well, the entire product will be ≤ 6 if the prodcut of $D_2 \ldots D_n$ is ≤ 6, which is given by p_{n-1} (because $D_2 \ldots D_n$ has $(n-1)$ dice). Now we know that

D_1	Probability of D_1	Probability of $D_2 \ldots D_n$
1	1/6	p_{n-1}

Now we go for the next easiest: if D_1 is $4, 5, 6$. If this is the case, then $D_2 \ldots D_n$ must all be 1, which has probability $\frac{1}{6^{n-1}}$. This adds three more rows to our table:

D_1	Probability of D_1	Probability of $D_2 \dots D_n$
1	1/6	p_{n-1}
4	1/6	$\frac{1}{6^{n-1}}$
5	1/6	$\frac{1}{6^{n-1}}$
6	1/6	$\frac{1}{6^{n-1}}$

If D_1 is 3, then there can be one of the dice $D_2 \dots D_n$ that is 2 and the rest by 1, or they can all be 1. Because any one of them can be a 2, we have that the probability that one of them is a two is $\frac{n-1}{6^{n-1}}$. The probability that all of them is one is $\frac{n-1}{6^{n-1}}$. Adding these two probabilities together, we have a new entry on our table

D_1	Probability of D_1	Probability of $D_2 \dots D_n$
1	1/6	p_{n-1}
3	1/6	$\frac{n}{6^{n-1}}$
4	1/6	$\frac{1}{6^{n-1}}$
5	1/6	$\frac{1}{6^{n-1}}$
6	1/6	$\frac{1}{6^{n-1}}$

If D_1 is 2, then there can be one of the dice $D_2 \dots D_n$ that is 3 and the rest by 1, or there can be one of the dice $D_2 \dots D_n$ that is 2 and the rest by 1, or they can all be 1. Because any one of them can be a 3, we have that the probability that one of them is a three is $\frac{n-1}{6^{n-1}}$. Because any one of them can be a 2, we have that the probability that one of them is a two is $\frac{n-1}{6^{n-1}}$. The probability that all of them is one is $\frac{n-1}{6^{n-1}}$. Adding these three probabilities together, we get our final probability:

D_1	Probability of D_1	Probability of $D_2 \dots D_n$
1	1/6	p_{n-1}
2	1/6	$\frac{2n-1}{6^{n-1}}$
3	1/6	$\frac{n}{6^{n-1}}$
4	1/6	$\frac{1}{6^{n-1}}$
5	1/6	$\frac{1}{6^{n-1}}$
6	1/6	$\frac{1}{6^{n-1}}$

Multipyling through our table, we get that

$$p_n = \frac{1}{6}p_{n-1} + \frac{3n+2}{6^n}$$

which gives us p_n in terms of probabilities before it.

2 Problem G2

We need to find a polynomial $a_k n^d + a_{k-1}n^{d-1} + \cdots + a_0$ equals $n + \frac{1}{n}$ for all integer n 1 through 99. This means we have to find all the coefficients such that the polynomial evaluates to $n + \frac{1}{n}$ for all integer n 1 through 99. We can accomplish this using a matrix. Before we start using a matrix, we rearrange the equation (by multiplying by n) to become

$$a_{98}n^{d+1} + a_{97}n^d + \cdots + a_0 n = n^2 + 1$$

$$\begin{bmatrix} 1 & 1 & 1 & \ldots \\ 2 & 4 & 8 & \ldots \\ 3 & 9 & 27 & \ldots \\ \vdots & \vdots & \ddots & \ldots \\ 99 & 9801 & 99^3 & \ldots \end{bmatrix} \begin{bmatrix} a_{98} \\ a_{97} \\ a_{96} \\ \vdots \\ a_0 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 10 \\ \vdots \\ 9802 \end{bmatrix}$$

Using the Java code here that I made with help from `https://stackoverflow.com/questions/19940740/gaussian-elimination-java` and `https://stackoverflow.com/questions/5061912/printing-out-a-2-d-array-in-matrix-format`

```java
package rref;

public class rrefMatrixRunner {

  public static void main(String[] args) {
    // TODO Auto-generated method stub
    int n = 99;
    double[][] matrix = new double[n][n+1];
    for (int i = 1; i<=n; i++) {
      for (int j = 1; j<=n+1; j++) {
        if (j!=n+1) {
          matrix[i-1][j-1] = Math.pow(i, j);
        }
        else {
          matrix[i-1][j-1] = Math.pow(i, 2)+1;
        }
      }
    }
    matrix = rref(matrix);

//     Print out last column of matrix
    for (int i = 0; i < matrix.length; i++) {
```

```
        System.out.print(matrix[i][matrix.length] + " ");
        System.out.println();
    }
  }


  public static double[][] rref(double[][] mat) {
    // TODO Auto-generated method stub
    double[][] rref = new double[mat.length][mat[0].length];

      /* Copy matrix */
      for (int r = 0; r < rref.length; ++r)
      {
          for (int c = 0; c < rref[r].length; ++c)
          {
              rref[r][c] = mat[r][c];
          }
      }

      for (int p = 0; p < rref.length; ++p)
      {
          /* Make this pivot 1 */
          double pv = rref[p][p];
          if (pv != 0)
          {
              double pvInv = 1.0 / pv;
              for (int i = 0; i < rref[p].length; ++i)
              {
                  rref[p][i] *= pvInv;
              }
          }

          /* Make other rows zero */
          for (int r = 0; r < rref.length; ++r)
          {
              if (r != p)
              {
                  double f = rref[r][p];
                  for (int i = 0; i < rref[r].length; ++i)
                  {
                      rref[r][i] -= f * rref[p][i];
                  }
              }
```

```
            }
        }
        return rref;
    }

}
```

which gives us

```
3.690402238709814
-5.0037792924052304
5.803434694717047
-3.7729943670966244
1.7650679308807378
-0.6192322701332906
0.16726044344451077
-0.035351754859144865
0.005893526335482517
-7.747085293925948E-4
7.93816390846828E-5
-6.142768188926767E-6
3.299241050681904E-7
-8.73820950462066E-9
-3.0295669303628555E-10
4.30028646044378E-11
-1.7546397737177478E-12
-6.148543586261023E-15
4.0084729242803866E-15
-1.668360207674011E-16
-2.201497923583278E-19
2.972780634580301E-19
-1.3390057655824836E-20
2.766685095124642E-22
-1.2930958744573467E-24
-1.3143562388346088E-25
8.63229571250134E-27
-3.3976490748046817E-28
6.434824831958213E-30
1.9028764026503805E-32
-2.6223280283851725E-33
2.6982883428578635E-35
-8.186588890534728E-37
3.733093875214879E-38
5.440253571137483E-40
```

```
-4.3200752119804546E-41
-2.1676936581706995E-44
1.4954324487806097E-44
6.242265652817103E-46
-2.9241162328792473E-47
2.4273261018895993E-49
3.1088229761338143E-51
-5.9356452412071515E-53
2.775650280760614E-54
-1.009403847188718E-55
1.3348645497755783E-57
-1.5566865897501515E-59
3.454332046516716E-61
4.890100176337677E-63
-3.4537045876753665E-64
5.514674703214455E-66
-6.88210871594159E-68
6.950182289622384E-70
1.5310395657631272E-71
-2.808240684817008E-73
-4.357637841139575E-75
1.055519761768652E-77
2.6813695624088435E-78
-1.6721739490463998E-80
-4.833742697431606E-82
8.110754464612074E-84
-8.578913728811781E-86
-6.3968668975154E-88
4.4444434482310236E-89
-2.436059455733525E-91
-5.906193656696549E-93
6.727235281727709E-95
-6.263956993883252E-97
-2.8728604441036385E-100
4.2300293876204653E-100
-6.949601110170572E-102
5.158256674514156E-104
-6.827227890117834E-106
4.5233591017877836E-108
1.0315554916480357E-109
-3.0173070668023568E-111
6.367414572996103E-113
-6.868182120608893E-115
```

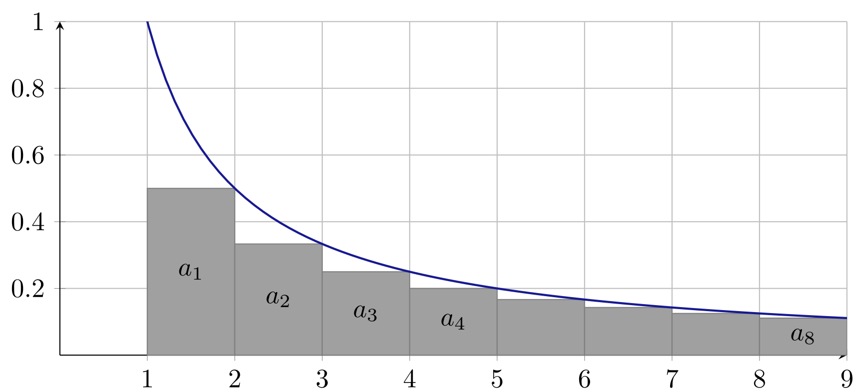
```
-2.4512527863912375E-117
1.0864671486725362E-118
-6.432957606644309E-121
-1.220695802531799E-123
5.3733247536635E-125
-6.834162133520749E-128
-1.7227898627020062E-128
1.4856908130282316E-130
1.4592521887207792E-132
-8.948993243975215E-135
-2.5041094126184992E-136
1.4056461985267213E-138
1.5137958091903056E-140
-5.710326474371304E-144
-2.3248321454570663E-144
1.9514381571279865E-146
-1.3810407145211485E-148
1.7058916267753447E-150
-1.3969606479992607E-152
5.514137182970936E-155
-8.422307765415783E-158
```

3 Problem M2

Any sum

$$\sum_{n=2}^{\infty} a_n$$

can be represented as a graph like this:



where the area of the nth gray rectangle equals a_n. Thus, if the integral from 2 to infinty converges (the area under the curve), so does the sum (the areas of the gray rectangles under the curve). To find the convergence of

$$\sum_{n=2}^{\infty} \ln(x^3+k) - \ln(x^3-k)$$

we first find the convergence of

$$\int_2^{\infty} \ln(x^3+k) - \ln(x^3-k)dx$$

Using WolframAlpha, we get that this integral is

$$-\frac{1}{2}\sqrt[3]{k}\ln[k^{2/3}+\sqrt[3]{k}x+x^2] - \frac{1}{2}\sqrt[3]{k}\ln[k^{2/3}-\sqrt[3]{k}x+x^2] - x\ln[x^3-k] + x\ln[x^3+k] + \sqrt[3]{k}\ln[\sqrt[3]{k}-x] + \sqrt[3]{k}\ln[\sqrt[3]{k}+x] - \sqrt{3}\sqrt[3]{k}\arctan\left[\frac{\sqrt[3]{k}+2x}{\sqrt{3}\sqrt[3]{k}}\right] + \sqrt{3}\sqrt[3]{k}\arctan\left[\frac{2x-\sqrt[3]{k}}{\sqrt{3}\sqrt[3]{k}}\right]$$

Simplifying using log rules,

$$\sqrt[3]{k}\ln\left(\frac{[\sqrt[3]{k}+x][\sqrt[3]{k}-x]}{\sqrt{k^{2/3}+\sqrt[3]{k}x+x^2}\sqrt{k^{2/3}-\sqrt[3]{k}x+x^2}}\right) + x\ln\left(\frac{x^3+k}{x^3-k}\right) + \sqrt{3}\sqrt[3]{k}\left(\arctan\left[\frac{2x-\sqrt[3]{k}}{\sqrt{3}\sqrt[3]{k}}\right] - \arctan\left[\frac{\sqrt[3]{k}+2x}{\sqrt{3}\sqrt[3]{k}}\right]\right)$$

Evaluating at infinity, this becomes zero (as seen by the application of l'Hopital's rule). Evaluating at two, we get

$$\sqrt[3]{k}\ln\left(\frac{[\sqrt[3]{k}+2][\sqrt[3]{k}-2]}{\sqrt{k^{2/3}+\sqrt[3]{k}2+2^2}\sqrt{k^{2/3}-\sqrt[3]{k}2+2^2}}\right) + 2\ln\left(\frac{2^3+k}{2^3-k}\right) + \sqrt{3}\sqrt[3]{k}\left(\arctan\left[\frac{4-\sqrt[3]{k}}{\sqrt{3}\sqrt[3]{k}}\right] - \arctan\left[\frac{\sqrt[3]{k}+4}{\sqrt{3}\sqrt[3]{k}}\right]\right)$$

which is a finite number; thus we conclude that the original sum converges.

For part two, we need to evaluate the sum at $k = 1$. We can rewrite the sum as

$$\ln\left(\prod_{n=2}^{\infty}\frac{x^3+1}{x^3-1}\right)$$

We can split the product into two products:

$$\ln\left(\prod_{n=2}^{\infty}\frac{x+1}{x-1}\right) + \ln\left(\prod_{n=2}^{\infty}\frac{x^2-x+1}{x^2+x+1}\right)$$

These are both telescopic products! We can rewrite them as

$$\ln\left(\prod_{n=2}^{\infty}\frac{(x+2)-1}{x-1}\right)+\ln\left(\prod_{n=2}^{\infty}\frac{(x-1)^2+(x-1)+1}{x^2+x+1}\right)$$

Plugging in numbers makes this pattern more obvious;

$$\ln\left(\prod_{n=2}^{\infty}\frac{3}{1}\frac{4}{2}\frac{5}{3}\frac{6}{4}\right)+\ln\left(\prod_{n=2}^{\infty}\frac{3}{7}\frac{7}{13}\frac{13}{21}\right)$$

Simplifying, we get that the sum is equal to

$$\ln\left(\frac{3}{2}\right)\approx 0.405$$