The Reciprocal of the Primes

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Before we go to primes, we first must visit several other sums, the first of which, is

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

1 Reciprocal of Natural Numbers

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

We can see that the 9 numbers (1 to 1/9) are each greater than 1/10. Thus, adding up the numbers (1 to 1/9) are definitely greater than $9 \cdot \frac{1}{10} = \frac{9}{10}$. We can repeat this logic for numbers (1/10 to 1/99), reasoning that all ninety of these numbers are greater than 1/100, thus the sum of these 90 numbers is greater than 90/100 = 9/10. We can use this logic with the numbers (1/100 to 1/999) and so on, so that we can write

$$\sum_{n=1}^{\infty} \frac{1}{n} > \underbrace{\frac{9}{10}}_{1 \dots \frac{1}{9}} + \underbrace{\frac{9}{10}}_{\frac{1}{10} \cdots \frac{1}{99}} + \dots = \infty$$

We can see that we are approaching infinity at the speed of a logarithm (from our number ranges of 1/10 to 1/100 and so on with powers of ten), so we can write our sum as

2 Reciprocals of Squares

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

A good way to evaluate this famous sum (see Basel Problem) is to bring back our old friend, the digamma function, ψ.

$$\psi'(z) = \frac{d^2}{dz^2} \ln(\Gamma(z)) = \sum_{n=0}^{\infty} \frac{1}{(n+z)^2}$$

We can let $z = 1 - z$ and $n = -n - 1$ to change our sum to be

$$\psi'(1-z) = \frac{d^2}{dz^2} \ln(\Gamma(1-z)) = \sum_{n=-1}^{-\infty} \frac{1}{(-n-1+1-z)^2} = \sum_{n=-1}^{-\infty} \frac{1}{(n+z)^2}$$

Adding the sums together, we get

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+z)^2} = \frac{d^2}{dz^2} \ln(\Gamma(1-z)) + \frac{d^2}{dz^2} \ln(\Gamma(z))$$

The sum of the derivatives is the derivative of the sum, so

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+z)^2} = \frac{d^2}{dz^2} \ln(\Gamma(1-z)) + \ln(\Gamma(z))$$

$$= \frac{d^2}{dz^2} \ln(\Gamma(1-z)\Gamma(z)) = \frac{d^2}{dz^2} \ln\left(\frac{\pi}{\sin(\pi z)}\right) = \frac{\pi^2}{\sin^2 \pi z}$$

Subtracting $1/z^2$ from both sides and taking the limit as $z \to 0$

$$2\sum_{n=1}^{\infty} \frac{1}{n^2} = \lim_{z\to 0} \left(\frac{\pi^2}{\sin^2 \pi z} - \frac{1}{z^2}\right) = \frac{\pi^2}{3}$$

meaning that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

3 Reciprocals of Primes

Before we talk about primes, we will talk about another way of writing $\sum_{n=1}^{\infty} \frac{1}{n^s}$. We can sieve out all thr primes by doing the following: we factor out all the multiples of two

$$\left(1 - \frac{1}{2^s}\right) \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \dots$$

Then we take out all the multiples of three

$$\left(1 - \frac{1}{3^s}\right) \left(1 - \frac{1}{2^s}\right) \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \dots$$

If we do this for all the primes, sieving away all the numbers prime by prime, we get that

$$\left(\prod_p 1 - \frac{1}{p^s}\right) \sum_{n=1}^{\infty} \frac{1}{n^s} = 1$$

meaning that

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}}$$

setting $s = 1$, we get

$$\sum_{n=1}^{\infty} \frac{1}{n} = \prod_p \frac{1}{1 - \frac{1}{p}}$$

Taking the log of both sides,

$$\ln \left(\sum_{n=1}^{\infty} \frac{1}{n}\right) = \sum_p \ln \left(\frac{1}{1 - \frac{1}{p}}\right)$$

and simplifying

$$\ln \ln \infty = -\sum_p \ln \left(1 - \frac{1}{p}\right)$$

The Taylor series of $\ln(1-x)$ is

$$\ln(1-x) = -x - \frac{x^2}{2} - \ldots - \frac{x^n}{n} - \ldots$$

meaning that

$$-\sum_p \ln\left(1-\frac{1}{p}\right) = \sum_p \left(\frac{1}{p} + \frac{1}{2p^2} + \frac{1}{3p^3} + \ldots\right) = \sum_p \frac{1}{p} + \frac{1}{2}\sum_p \frac{1}{p^2} + \frac{1}{3}\sum_p \frac{1}{p^3} + \ldots$$

We denote

$$S = \frac{1}{2}\sum_p \frac{1}{p^2} + \frac{1}{3}\sum_p \frac{1}{p^3} + \ldots$$

We know that

$$\frac{1}{2}\sum_p \frac{1}{p^2} > \sum_p \frac{1}{p^3}$$

because 2 is the smallest prime, so $2p^2 \leq p^3$, so $\frac{1}{2p^2} \geq \frac{1}{p^3}$. Multiplying both sides by 1/3, we get $\frac{1}{6p^2} \geq \frac{1}{3p^3}$, and because $1/4 > 1/6$, it is obvious that

$$\frac{1}{4}\sum_p \frac{1}{p^2} \geq \frac{1}{3}\sum_p \frac{1}{p^3}$$

In general,

$$\frac{1}{2^{n-1}}\sum_p \frac{1}{p^2} > \frac{1}{n}\sum_p \frac{1}{p^n}$$

Because

$$\frac{1}{2^{n-2}}\sum_p \frac{1}{p^2} \geq \sum_p \frac{1}{p^n}$$

and

$$\frac{1}{2} > \frac{1}{n}$$

Doing a term by term comparison, we see that

$$S < \frac{1}{2}\sum_p \frac{1}{p^2} + \frac{1}{4}\sum_p \frac{1}{p^2} + \frac{1}{8}\sum_p \frac{1}{p^2} + \frac{1}{16}\sum_p \frac{1}{p^2} + \dots$$

This is just a geometric series! Simplifying the sum, we get

$$S < \sum_p \frac{1}{p^2}$$

which is obviously less than $\sum_{n=1}^{\infty} \frac{1}{n^2} - 1$, where the one is from $1/1$, because 2 is the smallest prime number, so

$$S < \frac{\pi^2}{6} - 1$$

Going back to the original expression, we get

$$\ln \ln \infty = \sum_p \frac{1}{p} + \text{tiny number}$$

WHICH MEANS THAT $\sum_p \frac{1}{p}$ APPROACHES ∞ AT THE SPEED OF $\ln \ln x$!!!