Stirling's Aproximation for the Factorial

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1 Factorials

Reminders: The factorial is defined as $n! = n \cdot (n-1) \cdots 2 \cdot 1$, and the gamma function is defined as $\int_0^{\infty} t^{x-1} e^{-t} \, d = \frac{x!}{x}$

1.1 The Infinite Product

The factorial function can also be represented as an infinite product. To start, we must accept that for any integer m,

$$\lim_{n \to \infty} \frac{n!(n+1)^m}{(n+m)!} = 1$$

This makes sense because both the top and the bottom have a n^m term. It turns out that this also holds for any complex number z.

$$1 = \lim_{n \to \infty} \frac{n!(n+1)^z}{(n+z)!}$$

Multiplying both sides by $z!$

$$z! = \lim_{n \to \infty} n! \frac{z!}{(n+z)!} (n+1)^z$$

Simplifying, we get

$$z! = \lim_{n \to \infty} \frac{1 \cdots n}{(1+z) \cdots (n+z)} (n+1)^z$$

Condensing into product notation, we get

$$z! = \lim_{n\to\infty} \left(\prod_{i=1}^{n} \frac{i}{i+z} \right) (n+1)^z$$

We can express $(n+1)^z$ as $\prod_{i=1}^{\infty} \frac{(n+1)^z}{n^z}$ which becomes obvious when we write out the product $\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \ldots \cdot \frac{(n+1)^z}{n^z}$ where everything cancels except for $\lim_{n\to\infty} (n+1)^z$. Condensing the product further we get

$$z! = \prod_{n=1}^{\infty} \frac{n}{n+z} \frac{(n+1)^z}{n^z}$$

We will write this as the equivalent expression

$$z! = \prod_{n=1}^{\infty} \frac{1}{1+\frac{z}{n}} \left(1+\frac{1}{n}\right)^z = \prod_{n=1}^{\infty} \frac{\left(1+\frac{1}{n}\right)^z}{1+\frac{z}{n}}$$

Using this definition, the Gamma Function can be written as

$$\Gamma(z) = \frac{1}{z} \prod_{n=1}^{\infty} \frac{\left(1+\frac{1}{n}\right)^z}{1+\frac{z}{n}}$$

1.2 The Digamma Function

The digamma function ψ is defined as the derivative of the log gamma function; $\frac{d}{dx} ln(\Gamma(x))$.

$$\ln(\Gamma(z)) = \ln \left(\frac{1}{z} \prod_{n=1}^{\infty} \frac{(1+\frac{1}{n})^z}{1+\frac{z}{n}} \right)$$

Using log rules, we get

$$-\ln(z) + \sum_{n=1}^{\infty} z \ln\left(1+\frac{1}{n}\right) - \ln\left(1+\frac{z}{n}\right)$$

Now taking the derivative with respect to z

$$-\frac{1}{z}+\sum_{n=1}^{\infty}\ln\left(1+\frac{1}{n}\right)-\frac{\frac{1}{n}}{1+\frac{z}{n}}$$

And simplifying we get

$$\psi=-\frac{1}{z}+\sum_{n=1}^{\infty}\ln\left(1+\frac{1}{n}\right)-\frac{1}{n+z}$$

For our purposes, the derivative of digamma function is more useful; ψ'

$$\psi'=\frac{1}{z^2}+\sum_{n=1}^{\infty}\frac{1}{(n+z)^2}=\sum_{n=0}^{\infty}\frac{1}{(n+z)^2}$$

Now, if we find an approximation to the sum of reciprocal of squares, we can approximate the gamma function.

1.3 Approximating ψ'

To approximate ψ', we will use $\sum\frac{1}{n^2}$ just for ease of writing and notation. We can write $\sum\frac{1}{n^2}$ as a sum of telescopic series, helping us approximate it.

$$\begin{aligned}\sum\frac{1}{n^2}&=\sum\frac{1}{n(n+1)}+\sum\frac{1}{n^2(n+1)}\\&=\sum\left(\frac{1}{n}-\frac{1}{n+1}\right)+\sum\frac{1}{n^2(n+1)}\end{aligned}$$

The next telescopic term is $\sum\left(\frac{1}{n^2}-\frac{1}{(n+1)^2}\right)=\sum\frac{2n+1}{n^2(n+1)^2}$. Now we can design the series for the leftover term from above: $\sum\frac{1}{n^2(n+1)}$. $\sum\frac{1}{n^2(n+1)}-\sum\frac{2n+1}{n^2(n+1)^2}=\sum\frac{n}{n^2(n+1)^2}$ doesn't work because there is an n left in the numerator, so we do $\sum\frac{1}{n^2(n+1)}-\frac{1}{2}\sum\frac{2n+1}{n^2(n+1)^2}=\frac{1}{2}\sum\frac{1}{n^2(n+1)^2}$

$$\sum\frac{1}{n^2(n+1)}=\frac{1}{2}\sum\frac{2n+1}{n^2(n+1)^2}+\frac{1}{2}\sum\frac{1}{n^2(n+1)^2}$$
$$\sum\frac{1}{n^2}=\sum\left(\frac{1}{n}-\frac{1}{n+1}\right)+\frac{1}{2}\sum\left(\frac{1}{n^2}-\frac{1}{(n+1)^2}\right)+\frac{1}{2}\sum\frac{1}{n^2(n+1)^2}$$

The next telescopic term is $\sum\left(\frac{1}{n^3}-\frac{1}{(n+1)^3}\right)=\sum\frac{3n^2+3n+1}{n^3(n+1)^3}$. Now we can design the series for the leftover term from above: $\frac{1}{2}\sum\frac{1}{n^2(n+1)^2}$. We will leave the $\frac{1}{2}$ out for now and add it back on later. $\sum\frac{1}{n^2(n+1)^2}-\frac{1}{3}\sum\frac{3n^2+3n+1}{n^3(n+1)^3}=-\frac{1}{3}\sum\frac{1}{n^3(n+1)^3}$. The $\frac{1}{3}$ was put there to cancel out the n^2 and n and just leave the constant term. After we bring back the one half, and simplify, we get

$$\sum\frac{1}{n^2}=\sum\left(\frac{1}{n}-\frac{1}{n+1}\right)+\frac{1}{2}\sum\left(\frac{1}{n^2}-\frac{1}{(n+1)^2}\right)+\frac{1}{6}\sum\left(\frac{1}{n^3}-\frac{1}{(n+1)^3}\right)-\frac{1}{6}\sum\frac{1}{n^3(n+1)^3}$$

The next telescopic term is $\sum\left(\frac{1}{n^4}-\frac{1}{(n+1)^4}\right)=\sum\frac{4n^3+6n^2+4n+1}{n^4(n+1)^4}$. However, $\sum\frac{1}{n^3(n+1)^3}-\sum\frac{4n^3+6n^2+4n+1}{n^4(n+1)^4}$ will always have an n^3 in the numerator, so we move on. From now on, T_a will signify $\sum\left(\frac{1}{n^a}-\frac{1}{(n+1)^a}\right)$ to save space and effort.

The next telescopic term is $T_5=\frac{5n^4+10n^3+10n^2+5n+1}{n^5(n+1)^5}$. Now we can design the series for the leftover term from above: $-\frac{1}{6}\sum\frac{1}{n^3(n+1)^3}$. We will leave the $-\frac{1}{6}$ out for now and add it back on later. $\sum\frac{1}{n^3(n+1)^3}-\frac{1}{5}\frac{5n^4+10n^3+10n^2+5n+1}{n^5(n+1)^5}=-\frac{1}{5}\sum\frac{5n^2+5n+1}{n^5(n+1)^5}$. We added the $\frac{1}{5}$ to cancel out the n^4 and n^3. After we bring back the $-\frac{1}{6}$, and simplify, we get

$$\sum\frac{1}{n^2}=T_1+\frac{1}{2}T_2+\frac{1}{6}T_3-\frac{1}{30}T_5+\frac{1}{30}\sum\frac{5n^2+5n+1}{n^5(n+1)^5}$$

Now that we have enough precision in our approximation for $\sum\frac{1}{n^2}$, we can clarify the bounds of the sum and evaluate T_a. We redefine $T_a=\sum_{n=z}^{\infty}\left(\frac{1}{n^a}-\frac{1}{(n+1)^a}\right)$, and using the folding properties of telescopic sums, we get that

$$\sum_{n=z}^{\infty}\frac{1}{n^2}\left(\text{aka}\sum_{n=0}^{\infty}\frac{1}{(n+z)^2}\right)=\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{6z^3}-\frac{1}{30z^5}+\dots$$

Which we can write in terms of ψ' as

$$\psi'=\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{6z^3}-\frac{1}{30z^5}+\dots$$

1.4 Integrating Back to Γ

We will use big-O notation from now on, which looks like this: $\psi' = \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \mathcal{O}\left(\frac{1}{z^5}\right)$ which means that there are higher order terms starting with some constant times $\frac{1}{z^5}$. Back to integration. Integrating with respect to z, we get

$$\psi\left(\text{aka}\frac{d}{dz}\ln(\Gamma(z))\right) = \ln(z) - \frac{1}{2z} - \frac{1}{12z^2} + \mathcal{O}\left(\frac{1}{z^4}\right) + c_1$$

Integrating again with respect to z

$$\ln(\Gamma(z)) = z\ln(z) - z - \frac{1}{2}\ln(z) + \frac{1}{12z} + \mathcal{O}\left(\frac{1}{z^3}\right) + c_1 z + c_2$$

Grouping terms together

$$\ln(\Gamma(z)) = \left(z - \frac{1}{2}\right)\ln(z) - z + \frac{1}{12z} + \mathcal{O}\left(\frac{1}{z^3}\right) + c_1 z + c_2$$

On the first page of this document, I wrote the defining characteristic of the gamma function: $\Gamma(z) = \frac{z!}{z}$ or in other words $\frac{\Gamma(z+1)}{\Gamma(z)} = z$. Now taking the logarithm of both sides, we get $\ln\Gamma(z+1) - \ln\Gamma(z) = \ln z$. Using the equation above, we get

$$\left((z+1) - \frac{1}{2}\right)\ln(z+1) - (z+1) + \frac{1}{12(z+1)} + \mathcal{O}\left(\frac{1}{z^3}\right) + c_1(z+1) + c_2$$
$$- \left(\left(z - \frac{1}{2}\right)\ln(z) - z + \frac{1}{12z} + \mathcal{O}\left(\frac{1}{z^3}\right) + c_1 z + c_2\right) = \ln z$$

And distributing the negatives and simplifying a bit, we get

$$\left(z + \frac{1}{2}\right)\ln(z+1) - z - 1 + \frac{1}{12(z+1)} + \mathcal{O}\left(\frac{1}{z^3}\right) + c_1 z + c_1 + c_2$$
$$- \left(z - \frac{1}{2}\right)\ln(z) + z - \frac{1}{12z} + \mathcal{O}\left(\frac{1}{z^3}\right) - c_1 z - c_2 = \ln z$$

And simplifying more,

$$\left(z+\frac{1}{2}\right)\ln(z+1)-\left(z-\frac{1}{2}\right)\ln(z)+\frac{1}{12(z+1)}-\frac{1}{12z}+\mathcal{O}\left(\frac{1}{z^3}\right)+c_1-1=\ln z$$

Subtracting the $\ln z$ from the right side (being extremely careful of the minus signs everywhere of course), and grouping terms together

$$\left(z+\frac{1}{2}\right)(\ln(z+1)-\ln(z))+\frac{1}{12(z+1)}-\frac{1}{12z}+\mathcal{O}\left(\frac{1}{z^3}\right)+c_1-1=0$$

Now as we take the limit as z goes to infinity, we get

$$\lim_{z\to\infty}\left(z+\frac{1}{2}\right)(\ln(z+1)-\ln(z))+c_1-1=0$$

Using log properties,

$$\lim_{z\to\infty}\ln\left(1+\frac{1}{z}\right)^{z+\frac{1}{2}}+c_1-1=0$$

And as one of the major definitions of $e=\lim_{z\to\infty}\left(1+\frac{1}{z}\right)^z$, the limit evaluates to

$$\ln e+c_1-1=1+c_1-1=c_1=0$$

Plugging in c_1

$$\ln(\Gamma(z))=\left(z-\frac{1}{2}\right)\ln(z)-z+\frac{1}{12z}+\mathcal{O}\left(\frac{1}{z^3}\right)+c_2$$

We still have the c_2 constant, which we will figure out using Legendre's Duplication Formula:

$$\Gamma(x)\Gamma\left(x+\frac{1}{2}\right)=\sqrt{\pi}2^{1-2x}\Gamma(2x)$$

2 Legendre's Duplication Formula

To prove the duplication formula, we first begin with our old friend, the gamma reflection formula.

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \tag{1}$$

Secondly, we establish an identity for a product of sines:

$$\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right) = \frac{n}{2^{n-1}} \tag{2}$$

The famous complex exponential formula gives us that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. If we use $-\theta$ instead of θ, we get $e^{-i\theta} = \cos(\theta) - i\sin(\theta)$. If we subtract the second from the first, we get $e^{i\theta} - e^{-i\theta} = 2i\sin(\theta) \to \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \sin(\theta)$ and plugging in $\theta = \frac{k\pi}{n}$, we get

$$\frac{1}{2i}(e^{i\frac{k\pi}{n}} - e^{-i\frac{k\pi}{n}}) = \sin(\frac{k\pi}{n})$$

Plugging into the product above

$$\prod_{k=1}^{n-1} \frac{1}{2i}(e^{i\frac{k\pi}{n}} - e^{-i\frac{k\pi}{n}})$$

Manipulating, we get

$$\frac{1}{2^{n-1}} \prod_{k=1}^{n-1} \frac{e^{i\frac{k\pi}{n}}}{i}(1 - e^{-2i\frac{k\pi}{n}})$$

We know from looking at roots of unity that

$$z^n - 1 = \prod_{k=1}^{n} (z - e^{2i\frac{k\pi}{n}})$$

where $e^{2i\frac{k\pi}{n}}$ are the roots of unity, just like $z^4 - 1 = (z-1)(z+1)(z-1)(z+i) = \prod_{k=1}^{4}(z - e^{2i\frac{k\pi}{4}})$ where $1, -1, i, -i$ are the fourth roots of unity. Also, note that $\prod_{k=1}^{n}(z - e^{2i\frac{k\pi}{n}}) = \prod_{k=1}^{n}(z - e^{-2i\frac{k\pi}{n}})$ due to the symmetry of the unit circle; one goes clockwise from zero, one goes counterclockwise. Rearranging the formula a bit, we get

$$z^n - 1 = \prod_{k=1}^{n-1} (z - e^{-2i\frac{k\pi}{n}}) \cdot (z - e^{-2i\pi}) \to z^n - 1 = \prod_{k=1}^{n-1} (z - e^{-2i\frac{k\pi}{n}}) \cdot (z - 1)$$

which we can write as

$$\frac{z^n - 1}{z - 1} = \prod_{k=1}^{n-1} (z - e^{-2i\frac{k\pi}{n}})$$

Taking the limit as $z \to \infty$, using L'Hopital's rule, we get

$$n = \prod_{k=1}^{n-1} (1 - e^{-2i\frac{k\pi}{n}})$$

Plugging in above, we get

$$\frac{n}{2^{n-1}} \prod_{k=1}^{n-1} \frac{e^{i\frac{k\pi}{n}}}{i}$$

$\displaystyle\prod_{k=1}^{n-1} \frac{e^{i\frac{k\pi}{n}}}{i}$ evaluates to

$$\frac{\exp(\sum_{k=1}^{n-1} \frac{i\pi}{n}(1 + 2 + 3 + \ldots + (n-1)))}{i^{n-1}}$$

which is just

$$\frac{\exp(\frac{i\pi}{n}\frac{n(n-1)}{2})}{i^{n-1}} = \frac{\exp(i\pi)(\frac{n-1}{2})}{i^{n-1}} = \frac{(-1)^{\frac{n-1}{2}}}{i^{n-1}} = \frac{i^{n-1}}{i^{n-1}} = 1$$

And finally, going back to the beginning, we get

$$\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right) = \frac{n}{2^{n-1}}$$

Now that we have all our identities written out, we will begin. We define a function $f(x)$

$$f(x) = \prod_{k=0}^{n-1} \Gamma\left(x + \frac{k}{n}\right) \tag{3}$$

Multiplying by x and manipulating, we get

$$\begin{aligned}
x\,f(x) &= x \prod_{k=0}^{n-1} \Gamma\left(x + \frac{k}{n}\right) \\
&= x\,\Gamma(x) \prod_{k=1}^{n-1} \Gamma\left(x + \frac{k}{n}\right) \\
&= \Gamma(x+1) \prod_{k=0}^{n-2} \Gamma\left(x + \frac{k+1}{n}\right) \\
&= \Gamma\left(x + \frac{(n-1)+1}{n}\right) \prod_{k=0}^{n-2} \Gamma\left(x + \frac{k+1}{n}\right) \\
&= \prod_{k=0}^{n-1} \Gamma\left(x + \frac{k+1}{n}\right) \\
&= \prod_{k=0}^{n-1} \Gamma\left(\left(x + \frac{1}{n}\right) + \frac{k}{n}\right) \\
&= f\left(x + \frac{1}{n}\right)
\end{aligned}$$

Changing the variable $x = \frac{x}{n}$, we get

$$\frac{x}{n} f\left(\frac{x}{n}\right) = f\left(\frac{x+1}{n}\right) \tag{4}$$

Multiplying $n^{x+1}\frac{1}{nf\left(\frac{1}{n}\right)}$ on both sides

$$xn^x f\left(\frac{x}{n}\right)\frac{1}{nf\left(\frac{1}{n}\right)} = n^{x+1} f\left(\frac{x+1}{n}\right)\frac{1}{nf\left(\frac{1}{n}\right)}$$

And if we set $G(x) = n^x f\left(\frac{x}{n}\right)\frac{1}{nf\left(\frac{1}{n}\right)}$, we get that

$$xG(x) = G(x+1)$$

Which is the gamma function identity! This hints that $G(x) = \Gamma(x)$. A fully rigorous proof of $G(x) = \Gamma(x)$ can be acheived through the Bohr-Mollerup Theorem, a theorem I will not prove in this paper. We can set

$$\Gamma(x) = n^x f\left(\frac{x}{n}\right)\frac{1}{nf\left(\frac{1}{n}\right)}$$

And manipulating, we get

$$f\left(\frac{x}{n}\right) = nf\left(\frac{1}{n}\right)\frac{\Gamma(x)}{n^x} \tag{5}$$

To find $f\left(\frac{1}{n}\right)$, we first find $f^2\left(\frac{1}{n}\right)$.

$$f^2\left(\frac{1}{n}\right) = \prod_{k=1}^{n-1}\Gamma\left(\frac{k}{n}\right)\Gamma\left(1-\frac{k}{n}\right)$$

where $\prod_{k=1}^{n-1}\Gamma\left(\frac{k}{n}\right) = \prod_{k=0}^{n-1}\Gamma\left(\frac{1}{n}+\frac{k}{n}\right)$ because $\Gamma((\frac{n}{n}) = \Gamma((1-\frac{0}{n}) = 1$ and $\prod_{k=1}^{n-1}\Gamma\left(\frac{k}{n}\right) = \prod_{k=1}^{n-1}\Gamma\left(1-\frac{k}{n}\right)$ because of symmetry. Using the reflection formula, (1), we get

$$f^2\left(\frac{1}{n}\right) = \prod_{k=1}^{n-1}\frac{\pi}{\sin(k\pi/n)}$$

Simplifying and using the product of sines, (2), we get

$$f^2\left(\frac{1}{n}\right) = \frac{1}{n}2^{n-1}\pi^{n-1}$$

Square rooting and multiplying by the n in (5)

$$f\left(\frac{x}{n}\right) = \sqrt{n2^{n-1}\pi^{n-1}}\frac{\Gamma(x)}{n^x}$$

and finally variable change $x = nx$, we get Gauss's multiplication formula, or the generalized version of the duplication formula

$$f(x) = \prod_{k=0}^{n-1}\Gamma\left(x+\frac{k}{n}\right) = \sqrt{n2^{n-1}\pi^{n-1}}\frac{\Gamma(nx)}{n^{nx}} \tag{6}$$

Setting $n = 2$, we get the duplication formula.

$$\Gamma(x)\Gamma\left(x+\frac{1}{2}\right) = \sqrt{\pi}2^{1-2x}\Gamma(2x)$$

3 The Grand Finale

Taking the natural log of the duplication formula

$$\ln\left(\Gamma(x)\right) + \ln\left(\Gamma\left(x+\frac{1}{2}\right)\right) = \ln\sqrt{\pi} + \ln(2^{1-2x}) + \ln(\Gamma(2x))$$

Taking our approximation of $\ln\Gamma(x)$ from Section 1

$$\ln(\Gamma(z)) = \left(z - \frac{1}{2}\right)\ln(z) - z + \frac{1}{12z} + \mathcal{O}\left(\frac{1}{z^3}\right) + c_2$$

and pluggin it in the log duplication formula, we get

$$\begin{aligned}\left(z - \frac{1}{2}\right)\ln(z) - z + \frac{1}{12z} + \mathcal{O}\left(\frac{1}{z^3}\right) + c_2 \\ + z\ln(z + \frac{1}{2}) - z - \frac{1}{2} + \frac{1}{12z+6} + \mathcal{O}\left(\frac{1}{z^3}\right) + c_2 \\ = \ln\sqrt{\pi} + \ln(2^{1-2x}) + \left(2z - \frac{1}{2}\right)\ln(2z) - z + \frac{1}{24z} + \mathcal{O}\left(\frac{1}{z^3}\right) + c_2\end{aligned}$$

Distributing and using log rules, we get

$$\begin{aligned}z\ln(z) - \frac{1}{2}\ln(z) - z + \frac{1}{12z} + \mathcal{O}\left(\frac{1}{z^3}\right) + c_2 \\ + z\ln(z + \frac{1}{2}) - z - \frac{1}{2} + \frac{1}{12z+6} + \mathcal{O}\left(\frac{1}{z^3}\right) + c_2 \\ = \ln\sqrt{\pi} + (1-2z)\ln(2) + \left(2z - \frac{1}{2}\right)\ln(2) + 2z\ln(z) - \frac{1}{2}\ln(z) - 2z + \frac{1}{24z} + \mathcal{O}\left(\frac{1}{z^3}\right) + c_2\end{aligned}$$

Simplifying,

$$z\ln(z + \frac{1}{2}) - z\ln(z) - \frac{1}{2} + \mathcal{O}\left(\frac{1}{z}\right) + c_2 = \ln\sqrt{\pi} + \frac{1}{2}\ln(2)$$

Simplifying more,

$$\ln\left(\frac{z + \frac{1}{2}}{z}\right)^z - \frac{1}{2} + \mathcal{O}\left(\frac{1}{z}\right) + c_2 = \ln\sqrt{2\pi}$$

And taking the limit as $z \to \infty$

$$\ln\left(e^{\frac{1}{2}}\right) - \frac{1}{2} + 0 + c_2 = \ln\sqrt{2\pi} \to c_2 = \ln\sqrt{2\pi}$$

And putting everything back together, we have

$$\ln(\Gamma(z)) = \left(z - \frac{1}{2}\right)\ln(z) - z + \frac{1}{12z} + \mathcal{O}\left(\frac{1}{z^3}\right) + \ln\sqrt{2\pi}$$

And exponentiating both sides,

$$\Gamma(z) = z^{z-\frac{1}{2}} \cdot e^{-z} \cdot \exp\left(\frac{1}{12z} + \mathcal{O}\left(\frac{1}{z^3}\right)\right) \cdot e^{\ln\sqrt{2\pi}}$$

And simplifying, we get

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}}\left(\frac{z}{e}\right)^z \exp\left(\frac{1}{12z} + \mathcal{O}\left(\frac{1}{z^3}\right)\right)$$

AND BECAUSE $z! = z\Gamma(z)$, WE GET

$$z! = \sqrt{2\pi z}\left(\frac{z}{e}\right)^z \exp\left(\frac{1}{12z} + \mathcal{O}\left(\frac{1}{z^3}\right)\right) \quad \square$$

and we are done.