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- 3blue1brown Linear Algebra ★★★!
- textbook (Axler's "Linear Algebra done right")

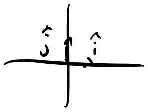
geometric intuition! V denote vector space, but is not a concrete example.

concrete examples: Most common / easy to visualize "vectorspaces"

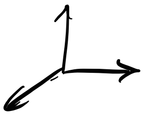
$\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$

(duality)

"real line"



"Euclidean plane"



"Euclidean space"

Set notation, basic proofs (contradiction)

$\boxed{V \text{ vector spaces}}$ $\boxed{\text{underlying set } V}$, plus some operations that satisfy some axioms.

$\boxed{(\underline{V}, +, \cdot, 1, 0, \mathbb{F})}$ usually $\mathbb{R}$.

group theory or ring theory — have underlying sets.
— different operation

G (G, •) abusing notation :(

"Element

" is in" or "is an element of"

$S := \{ x \in \text{some overarching set} : x \text{ satisfies some condition} \}$.

"defined as"
or " was defined as".

"such that"
others use the bar $|$.

$\{ \cdots : \cdots \}$
or $\{ \cdots \mid \cdots \}$

in English: "the set of all x in some overarching set such that x satisfies some condition".

Example: $S := \{ x \in \underbrace{\mathbb{Z}}_{\text{the set of integers}} : x = 2a$ ~~for some~~ $\exists\, a \in \mathbb{Z} \}$.

$\{ \cdots, -2, -1, 0, 1, 2, 3, \cdots \}$.

NOTATION

sidenote: in set theory, $\mathbb{N}$ usually denotes $\{0, 1, 2, 3, \cdots\}$, and I usually stick w/ this convention, reserving $\mathbb{N}^+ = \{1, 2, 3, \cdots\}$.

"blackboard bold"

$\mathbb{N}$ = set of natural numbers.
$= \{1, 2, 3, \cdots\}$.
$\mathbb{Z}$ = set of integers (German Zahlen)
$\mathbb{Q}$ = set of rationals (Q for quotient)
$\mathbb{R}$ = set of real #s
$\mathbb{C}$ = set complex #s $\quad \mathbb{R} + \mathbb{R}i = \{ a + bi : a, b \in \mathbb{R} \}$

in English: " set of all x in $\mathbb{Z}$ such that x is equal to 2a for some $a \in \mathbb{Z}$ }.

" set all integers x such that x is even "

5 elements of S: 10, 20, 30, 40, 50

~ "even #s".

NOTATION

$\forall$ "for all" or "for every"

$\exists$ "there is" or "there exists" or "for some".

Example

$$\boxed{S := \{ x \in \mathbb{Z} : \quad x = 2a \qquad \underset{\text{“for all”}}{\forall} a \in \mathbb{Z} \}.}$$

$= \emptyset$ “empty set”.

b/c there is no integer x such that it equals 2a for all $a \in \mathbb{Z}$.

$\Downarrow$ that would mean

$x = 2 \cdot 1 = 2$ $a = 1$

$= 2 \cdot 2 = 4$ $a = 2$

$= 2 \cdot 3 = 6$ $a = 3$

∴ not possible!

can't simultaneously equal 2 and 4 and 6!

Humorous example of ∀ and ∃:

true → ∀ people, ∃ a day that is their birthday.

false → ∃ a person, such that ∀ days are their birthday.
“all”

the only difference is order of ∀, ∃ vs. ∃, ∀.

Example:

$$S := \{ x \in \mathbb{R} : \underline{x = \tfrac{p}{q} \text{ for some } p, q \in \mathbb{Z}} \}.$$

5 elements of this set: $\frac{1}{2}$, $\frac{2}{3}$, — — not in there: $\frac{\pi}{4}$.

set of rational numbers, or “fractions”.

Functions $f(x) = x^2 + 2$.

Rigorous definition: 3 pieces of info:

- Domain : X (input space).
- Codomain : Y (output space)
- a ~~lot~~ set of pairs (x, y) where $x \in X$, $y \in Y$.

$\mathcal{P}$

satisfying 2 rules:

- every $x \in X$ has a corresponding $y \in Y$.

  (for every $x_0 \in X$, there exists some pair in $\mathcal{P}$ that looks like (x_0, y))

- for any pair (x, y), there is no other pair w/ the same first element.

Example: Let f denote the function:

- $\mathbb{R}$ domain
- $\mathbb{R}$ codomain
- the set of pairs $\{(x, x^2+2) : x \in \mathbb{R}\}$.

this is summarized in the sentence

"$f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \boxed{x^2+2}$".

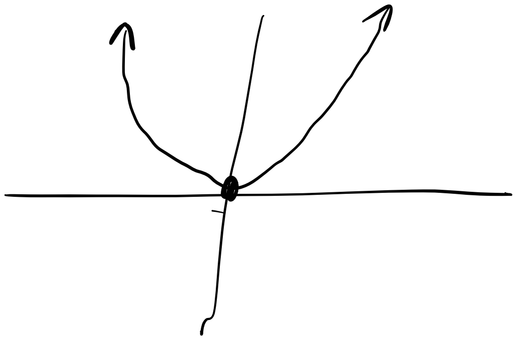
"$f: X \to Y$ defined - - - - -"

The 2 rules are satisfied:

- for every input x, I have an output $\boxed{x^2+2}$
- for any input x, I only have one output

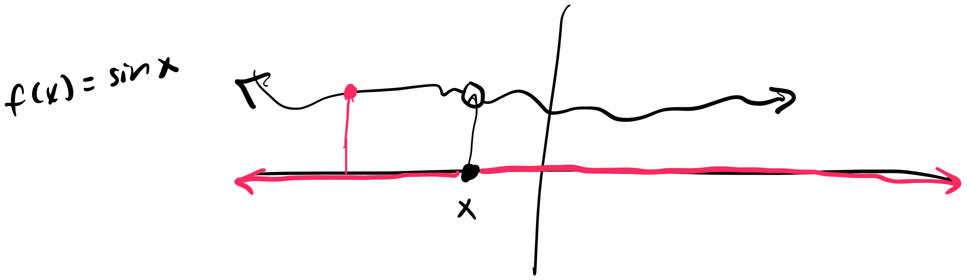
graph of functions.

$f(x)=x^2 : \mathbb{R} \to \mathbb{R}$



graph of $f:\boxed{\mathbb{R}}\to\mathbb{R}$ that does not satisfy rule 1

forbids holes in the graph.



fix: ① I can fill in the hole

or ② I can restrict the domain $\mathbb{R} - \{x\}$.

$f: \underbrace{\mathbb{R}-\{x\}} \to \mathbb{R}$.

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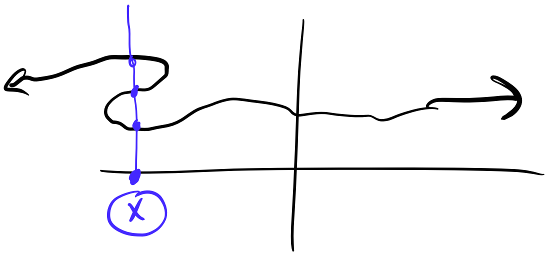
graph that does not satisfy rule 2

forbid graph crossing vertical lines more than once.



Main takeaway: when you talk about a function. you must not only define it using some formula

$f(x) = x^2 + 2$: $\mathbb{R} \to \mathbb{R}$

$\mathbb{Z} \to \mathbb{Z}$

$\mathbb{N} \to \mathbb{N}$

you **must** also specify the **domain** and **codomain**

$f(x) = -x$ $\qquad \mathbb{N} \to \mathbb{N}$ ✗ no!

$1 \mapsto -1$

$\mathbb{Z} \to \mathbb{Z}$ ✓

$\mathbb{N} \to \mathbb{Z}$.

by enlarging the codomain, I made it a well-defined function.

$S_1 := \{ x \in \mathbb{Z} : x = 2a$ for some $a \in \mathbb{Z} \}$.

$S_2 := \{ x \in \mathbb{Z} : x = 2a$ $\boxed{\text{for all } a \in \mathbb{Z}}$ $\}$.

way stronger than

your eyebrows should raise when you see "for all"!

the set of all even numbers.

is $2 \in S_2$? If 2 was in S_2, then

$2 = 2a$ for all $a \in \mathbb{Z}$.

this is false, b/c taking $a = -1$,

$2 = -2$, which is false.

6/23/22   115A

OH   2-3 pm   Tues to Thurs
      4-5pm   Tues

Point: when working w/ functions, always be sure to preferably write down but at the very least, think through what the domain & codomain of the function.

If the function's name is f, then write $f: \overset{\text{inputspace}}{A} \to \overset{\text{output}}{B}$

make sure that for $\overset{\exists}{\text{every}}$ input $x \in A$, f applied to the input x, denoted f(x) is a well-defined element of B.

uniqueness.

→ injectivity, surjectivity

Themes in math:

∃ "existential quantifier"

∀ "universal quantifier"

- existential statements
- uniqueness statements

Examples:

- existence uniqueness thm for ODE.

- intermediate value thm, if increasing, uniqueness.

Proofs

proof by contradiction.

Theorem there is no smallest positive rational.

K

① • suppose the contrary

in this that is: suppose there is a smallest positive rational.

• Goal: produce some kind of contradiction, or impossibility.

consider $\boxed{\frac{k}{2}}$.

- $\frac{k}{2}$ is rational. **Direct proof**: k rational exactly means that $k = \frac{p}{q}$ for $p, q \in \mathbb{Z}$.

  Then, $\frac{k}{2} = \frac{p}{2q}$. $p, 2q \in \mathbb{Z}$

  hence $\frac{k}{2} = \frac{p}{2q}$ is rational

- $\frac{k}{2}$ is positive. **Proof**: $k > 0$ dividing by a positive # 2 preserves inequality, so $\frac{k}{2} > \frac{0}{2} = 0$.

- $\frac{k}{2} < k$. **Proof**: $1 < 2$, and multiplication by positive # preserves inequality,

  $1 \cdot k < 2 \cdot k$, dividing $\frac{k}{2} < k$.

Very explicitly, the contradiction was that

1. k we said was smallest positive rational
2. we found a smaller positive rational.

contradict.

Theorem: a sum of a rational # & irrational # must be irrational.

Pf: by contradiction

① Suppose the contrary: we have a rational $\boxed{\frac{a}{b}}$ + a irrational # $\boxed{x}$ = a rational number $\boxed{\frac{c}{d}}$. $a, b, c, d \in \mathbb{Z}$.

$$\boxed{\frac{a}{b} + x = \frac{c}{d}.}$$

well, rearranging, we get

$-\frac{a}{b}$ $-\frac{a}{b}$

$$\boxed{x = \underbrace{\frac{c}{d} - \frac{a}{b}}_{\text{rational!}} = \frac{cb - ad}{bd}}$$

but $cb - ad \in \mathbb{Z}$

$bd \in \mathbb{Z}$.

Explicit contradiction:

1. we said x was irrational.
2. but our hypothesis lead to x being rational

Proof by induction:

Tip

Whenever you want to prove something for all $n \in \mathbb{N}^+$.
you will want to consider induction

How would one even go about proving infinitely many claims?

P(n)

Example: prove $1+2+\dots+n = \frac{n(n+1)}{2}$. for each $n \in \mathbb{N}$, there is one claim that can be numerically checked.

Thought experiment: suppose I had a very long line of dominoes.

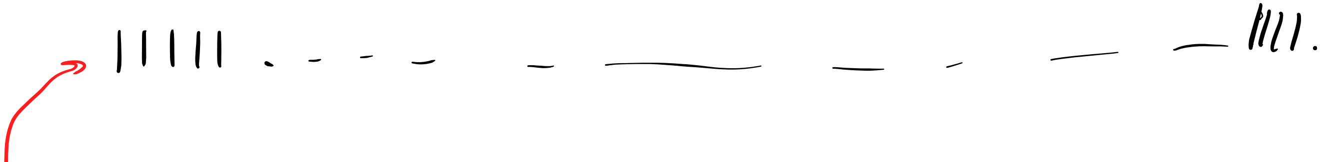
Scenario 1

I want you to push over the 1 billionth domino.

as drawn, you would have to travel all the way to the 1 billionth domino, and push it over.

what if

Scenario 2.



This now makes the assignment trivial! Just push over the 1st, and wait, and you are sure that the 1-billionth domino is knocked over.

The fundamental thing that changed is that in Scenario 2, I have given you that guarantee that every domino will knock over the next domino in line.

Principle of Induction.

↳ tells us if I can prove that P(n) implies P(n+1), then just by proving P(1), I will have proven P(N) for all $N \in \mathbb{N}$.

You give me any $N \in \mathbb{N}$ (a trillion), I knock over 1st domino, then wait! → Guaranteed that trillionth is knocked over.

Example: $1+2+\cdots+n \overset{\checkmark}{=} \frac{n(n+1)}{2}$. I want to prove $P(n) \Rightarrow P(n+1)$.

$P(n+1)$ is the claim that

$$\underbrace{1+2+\cdots+n}_{\text{"Induction hypothesis"}}+(n+1) \overset{?}{=} \frac{(n+1)(n+1+1)}{2} = \frac{(n+1)(n+2)}{2}.$$

$$\frac{n(n+1)}{2} + (n+1) = \frac{n(n+1)+2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$$

Just "knocking over" or "proving" P(1) finishes the entire theorem.

$P(1)$ says $1 \overset{?}{=} \frac{1(1+1)}{2}$ and indeed this is true.

Problem 10 of HW:

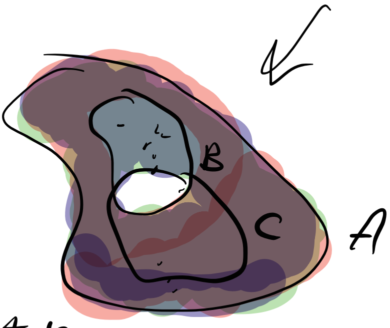
A set.

B, C subsets of A.

$\boxed{\text{WTS}}$ want to show. WTF want to find.

"de Morgan's law".

$$\boxed{A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)}$$



Draw the picture.

Pf:

Tip: when proving equality, prove 2 inequalities.

#s: $x = y \iff x \le y$ and $x \ge y$

sets: $X = Y \iff X \subseteq Y$, and $X \supseteq Y$

subset superset

Now we show $\boxed{LHS \subseteq RHS}$.

Suppose element $x \in LHS = \boxed{A \setminus (B \cap C)}$. Then, ~~$x \notin B$ or $x \notin C$~~

we want to show $x \in RHS$.

"initialization phrase" Then $x \notin B \cap C$. We have the following 2 cases:

(sadly it is not obvious that one "should" do the breakdown into these specific cases. But my belief is that such "leaps of thinking" come w/ experience ; time).

① If $x \in B$, then $x \notin C$ (otherwise contradict).

$\Downarrow$ "implies"

$x \in A \setminus C \Rightarrow x \in RHS$.

② otherwise $\boxed{x \notin B \Rightarrow x \in A \setminus B \Rightarrow x \in RHS.}$

$\boxed{RHS \subseteq LHS}$

Suppose $x \in RHS$. Then $x \in (A \setminus B) \cup (A \setminus C)$.

So 2 cases either $x \in A \setminus B \Rightarrow x \notin B \Rightarrow x \notin B \cap C$. ✓

or

$x \in A \setminus C \Rightarrow x \notin C \Rightarrow x \notin B \cap C$.

6/28/22

⓪ Proofs are for human communication. (ideals: concise and clear)

Tips:

① Good idea to always state the goal so,

(want to show)

WTS ..., we are asked to find ....

we want to determine ....

→ when you are proving subclaims

② use your words!

- if proof by contradiction / induction, say so at the beginning!
- initialization phrases "Suppose" "Let" "Consider"
- when breaking into cases, be clear about this! Best to say # of cases.
- connecting phrases like "we get", "which yields", "which produces". "hence" "and so" "then" "therefore".

③ You can use symbols if they are very common and lead to a smooth speaking experience.

ALWAYS Good to pretend you are writing a proof to give a presentation in front of others.

Notes found at:

Bruin Learn Discussion Tab

1.2

a. Every VS contains zero vector.

True, see axioms. Unique zero vector!

cancellation thm

corollary 1 of Thm 1.1

b. —

c. in a VS, $\boxed{a, b \in \mathbb{F},\ x \in V}$ $\boxed{ax = bx \Rightarrow a = b.}$

False, take $x = \vec{0}$. Then $\forall a, b \in \mathbb{F}$, $ax = bx = 0$.

what if $x \neq \vec{0}$? Proceed by contradiction...

Suppose the contrary. for sake of contradiction that $a \neq b$.

$\underbrace{\text{for sake of contradiction}}_{\text{fsoc.}}$

toward a contradiction.

Then $\underbrace{a-b}_{\in \mathbb{F}} \neq 0$, and hence has a multiplicative inverse $(a-b)^{-1}$.

Then $\underset{(a-b)x}{\underset{=}{ax = bx \Rightarrow ax - bx}} = 0$. Multiply on both sides by $(a-b)^{-1}$ (also known as "dividing by $(a-b)$")

we get $\underbrace{(a-b)^{-1}(a-b)}_{1} x = (a-b)^{-1} \cdot 0 = 0$

and so $x = 0$, ⨯ contradiction.

Tip for math: if want to prove nice thm, but easy counterexample, don't give up hope yet, merely restrict so ↑ no longer applies.

$a \in \mathbb{F}$, $x, y \in V$

d) in a VS $ax = ay \Rightarrow x = y$.

false, take $a = 0$.

Again what if $a \neq 0$? is . . . true!

By the same argument as above: divide by a!

Main idea of both proofs was "divide"!

$\mathbb{Z}$ not a field! Math 210 . . .

"module theory"

What exactly do I mean?

Remember to specify VS, must specify addition; scalar mult structure!

1.3. Problem 1.

a. V is a VS, $W \subseteq V$ (subset) that is also a VS, is W a subspace of V?

False

b.

1.3.10. $W_2 := \{(a_1 \cdots a_n) \in \overset{V}{\overset{\|}{\mathbb{F}^n}} : \underline{a_1 + \cdots + a_n = 1}\}$. is not

a subspace.

$0 \in W$ closed under $+$ closed under $\cdot$

0 element of V

Easiest to check

(Example, if $\mathbb{F} = \mathbb{R}$, then

$(\frac{1}{n}, \cdots, \frac{1}{n}) \in W_2$.

$(1, 0, \cdots, 0) \in W_2$

$(0, 1, 0, \cdots 0) \in W_2$

$\underbrace{0_V}_{\text{0 element of } V = \mathbb{F}^n}$ is exactly ... $\underline{(0, 0, \cdots, 0)}$.

$$\begin{array}{r} (0, 0, \cdots, 0) \\ + \ (a_1, \cdots, a_n) \in \mathbb{F}^n \\ \hline (a_1, \cdots, a_n). \end{array}$$

— Not a subspace b/c fails

— Not closed under $+$:

$$\begin{array}{r} (1, 0, \cdots, 0) \in W_2 \\ + \ (0, 1, 0 \cdots 0) \in W_2 \\ \hline (1, 1, 0, \cdots 0) \notin W_2 \end{array}$$

— Not closed under $\cdot$:

$$2 \cdot \underbrace{(1, 0, 0, \cdots 0)}_{\in W_2} = \underbrace{(2, 0, 0, \cdots, 0)}_{\notin W_2}.$$

P7 on the HW1. Idea. To show something equals 0, show it satisfies the properties that 0 satisfies. "quacks like a duck" principle

So WTS $a \cdot \vec{0} = \vec{0}$. So show $a \cdot \vec{0} + v = v$ for all $v \in V$. (idea). Turns out not to be best idea, but keep principle in mind

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Better idea. copy the proof of Thm 1.2 (a).

Why want to strengthen V4? i.e. why want uniqueness of additive

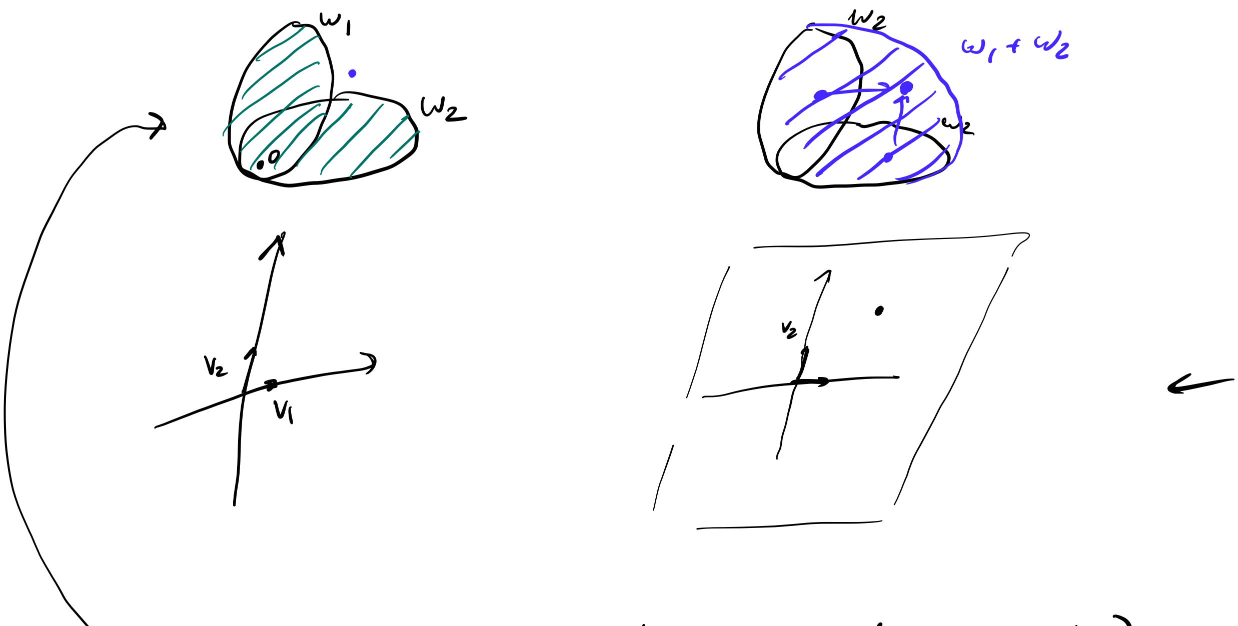
Suppose vector v, and it has 5 additive inverses

$w_1\ w_2\ w_3\ w_4\ w_5$.

we can not use the notation "$-v$" ?

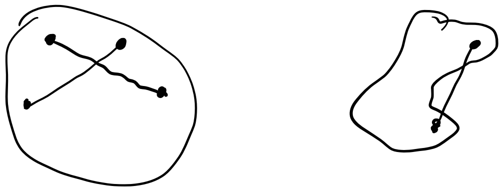
[The point of the thm (uniqueness of inverse) is that it allows to unambiguously use the notation "$\underline{-v}$".

$W_1 \cup W_2$ vs $W_1 + W_2$

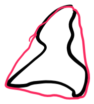


← concrete example in $\mathbb{R}^2$

analogy b/t vector space and convexity?



"convex hull"



Exactly Exercise 1.3.23(b).

You've shown $W_1 + W_2$ is a subspace.

WTS $W_1 \subseteq W_1 + W_2$. $W_2 \subseteq W_1 + W_2$ will follow by symmetry.

$\uparrow$

suppose $v \in W_1$. WTS $v \in W_1 + W_2$.

we know $0 \in W_2$. so $\underbrace{v}_{\in W_1} + \underbrace{0}_{\in W_2} \in W_1 + W_2$.

But $v = v + 0$, so we indeed have shown $v \in W_1 + W_2$.

WTS even functions on $\mathbb{R}$ ~~form~~ VS.

Example 3 of textbook says $\mathcal{F}(\mathbb{R}, \mathbb{F})$ (functions) form VS.

so suffices to show form subspace of $\mathcal{F}(\mathbb{R}, \mathbb{F})$. Just need to check 3 things from thm 1.3.

- closed under addition.

Suppose f, g are even functions on $\mathbb{R}$. WTS $f+g$ is even.

That means WTS $(f+g)(t) = (f+g)(-t) \quad \forall t \in \mathbb{R}$.

well,

$$(f+g)(t) := f(t) + g(t) \underset{\text{evenness of } f,g}{=} f(-t) + g(-t) =: (f+g)(-t)$$

Done. ✓

6/30/22

1.3.1

(a) V is a VS, $\underset{\text{subset}}{W \subseteq V}$ that is also a VS, is W a subspace of V?

What exactly do I mean?

Remember
to specify VS, mult.
specify addition ;
scalar
mult
structure!

False

If however W is subset of V that is also a VS w.r.t. (with respect to) the same + and scalar multiplication as V, then W is a subspace of V. (by definition, see §1.3 in book).

Counterexample

Consider the VS $\mathbb{R}$, w/ standard $\overset{\text{addition}}{+}$, $\overset{\text{scalar mult}}{\cdot}$.

($\mathbb{R}$ is a VS over the field $\mathbb{R}$)

consider the subset $(0, \infty)$

- not subspace of $\mathbb{R}$ b/c ... not closed under scaler mult.

easiest to check

Recall check subspace, check
- 0 in the
- closed under +
- closed under scalar mult.

However, let me equip $(0,\infty)$ w/ the $\mathbb{R}$-VS-structure

- $a \oplus b := ab.$
- $\underset{\in\mathbb{R}}{\lambda} \odot c := c^{\lambda}.$

Left for you to check VS axioms, however I'll do distributativity to show you:

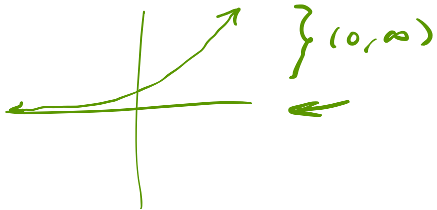
$$\lambda \odot (a \oplus b) \overset{?}{=} (\lambda \odot a) \oplus (\lambda \odot b)$$

$$\| \qquad\qquad\qquad //$$

$$(ab)^{\lambda} = a^{\lambda} b^{\lambda}$$

Seems a bit magical ... but I tell you it's actually the "same" VS as $(\mathbb{R}, +, \cdot)$!

In the following sense:



$\mathbb{R} \ni x \longmapsto e^x \in (0,\infty)$.

$x \longmapsto e^x$

$+ \longrightarrow$

$\cdot$

$\underline{+}$: $a+b \xrightarrow{x \mapsto e^x} e^{a+b} = e^a e^b = e^a \oplus e^b$

in other words, addition $\xrightarrow[\text{exponential map}]{}$ multiplication

$\underline{\cdot}$: $\lambda c \xrightarrow{x \mapsto e^x} e^{\lambda c} = (e^c)^\lambda = \lambda \odot e^c$.

Conclusion, $[x \mapsto e^x]$ exponential function is a VS isomorphism!

between $(\mathbb{R}, +, \cdot)$, and $((0,\infty), \oplus, \odot)$.

Bases, LI, span.

just a set of vectors inside VS V.

The point of a basis, B is that I want to represent every vector in V uniquely as a linear combination of elements of the basis

Recall above mentioned themes of math:

spanning [• Existential statement: $\exists$ a representation for every vector

linearly indep [• Uniqueness statement: unique linear combo.

Bases are therefore the spanning and linearly independent sets.

Def – spanning set S : every vector $v \in V$ can be written as linear combo of elements of S.

In other words, the elements of S (and their scalings sums :)) reach / span all of V.

- LI set $\{v_1 --- v_n\}$, if $v = a_1 v_1 + --- + a_n v_n$ and $v = b_1 v_1 + \dots + b_n v_n$. then $a_i = b_i \ \forall i$.

Special case

In particular, 0 vector has unique representation

$$0 = 0v_1 + ---- + 0v_n$$
$$= b_1 v_1 + --- + b_n v_n ,$$

then $b_i = 0$.

i.e. no nontrivial representation of $\vec{0}$.

↳ implies the full case, because ... if $a_1 v_1 + --- + a_n v_n = v = b_1 v_1 + --- + b_n v_n$

then $(a_1 - b_1) v_1 + ---- + (a_n - b_n) v_n = \vec{0}$.

$\Rightarrow a_i - b_i = 0 \ \forall i \iff a_i = b_i \ \forall i$.

---

why are bases useful.

- usually ∞ many vectors in V. However, usually basis is finite. so simplifies # of things we need to keep track of.
- especially useful for defining linear transformations. B/c

I can define $T: V \to W$ just by specifying value of T on basis of V.

- let's say basis $\{v_1 \cdots v_n\}$ of V, and we demand $T(\underset{\in V}{v_i}) = \underset{\in W}{w_i}$. I can then define T on all of V as follows. Let $v \in V$ be arbitrary. Then by basis we know $v = a_1 v_1 + \cdots + a_n v_n$ for some $a_i \in \mathbb{F}$.

"extension by linearity."

then, define

$$T(v) := a_1 T(v_1) + \cdots + a_n T(v_n)$$
$$= a_1 w_1 + \cdots + a_n w_n$$

- well-defined b/c is unique!

you can imagine something bad would happen if

Example of conspiracy.

$$v = a_1 v_1 + \cdots + a_n v_n \quad \text{but} \quad a_1 w_1 + \cdots + a_n w_n$$
$$= b_1 v_1 + \cdots + b_n v_n \qquad \neq b_1 w_1 + \cdots + b_n w_n$$

Different a_i, b_i conspire to yield same linear combination output.

cuz then what do I say $T(v)$ is?

- In other words, V is freely generated by basis

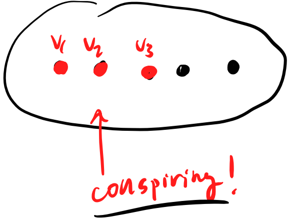
no conspiracies amongst the basis vectors.

spanning

One last thing to discuss, Thm 1.9. in book.

Thm 1.9 if VS V is generated by a finite set S (i.e. finite set S spans V), then some subset of V is basis for V (in particular, V has finite basis).

Example



Then, subset, say kick out v_3 to remove conspiracy.

In the sense that $v_3 = 2v_1 + 5v_2$.

2 ways of proving this : “Growing method” can start small (start w/ something you know is LI) and grow, till it's spanning.
(preserving LI)

- “Shrinking method” start big (i.e. w/ spanning set S) and remove until it's LI.
(preserving spanning).

Your ~~book~~ professor does “growing method”. (Monday this week?).

I will present the “shrinking method”.

- start w/ S, which is spanning. A is either LI or LD

If LI : done! Exactly basis.

If LD, then by def we have nontrivial rep of 0:

$$\sum_{\underset{\uparrow}{i\in I}} a_i v_i = 0 \quad \text{where} \quad a_i \neq 0 .$$

set of indices

example $I := \{1,3,4,5, 9, 17\}$.
when $\{v_1, \ldots, v_{20}\} = S$.

Then we can rearrange $a_j v_j = -\sum_{\substack{i\in I \\ i\neq j}} a_i v_i$

b/c not zero, can divide

$$\boxed{v_j} = \boxed{-\frac{1}{a_j} \sum_{\substack{i\in I \\ i\neq j}} a_i v_i}$$

Thus, v_j not necessary to span V, b/c if I have $v \in V$ written as

$$v = \sum_{i\neq j} a_i v_i + a_j \underbrace{v_j}$$

and get linear combo not involving v_j.

Thus, I can remove v_j from S and remain a spanning set.

In summary, I've shown:

- if S is LI, done.
- if S is LD, can remove one vector and still have a spanning set.

Iterate. B/c S is finite, and size is going down by 1 every step we run, will terminate in finitely many steps.

How is it possible to terminate? well, if LI ⇝ terminate.

if LD, not done, can continue removing.

So the only that the procedure could have terminated is if we ended w/ a LI set.

■

7/5/22

My sincerest apologies, my scheduling got a bit off due to the holiday yesterday, so I would like to reschedule today's discussion to 4-5 pm LA time. Now (1-2pm LA time) we can treat as office hours. Feel free to stay for OH and ask any questions you'd like to ask. (+ OH from 2-3 pm LA time)

Discussion **Agenda**:

- a theorem about LI sets being smaller than spanning sets (using the "replacement lemma"), and using this to prove all bases have the same size.

OH 7/5/22

HW2 problem 1 is I think HW 1 §1.8 #23 (a)

HW2 Problem 2 V is VS, $\underbrace{S_1, S_2}_{\boxed{\text{LI sets}}} \subseteq V$ s.t. $S_1 \cap S_2 = \emptyset$

WTS $S_1 \cup S_2$ is LD $\Leftrightarrow$ span $(S_1) \cap$ span $(S_2) \neq \{0\}$

↑ can replace w/ $\supsetneq$

Recall previous tip: prove equality by proving 2 inequalities, in this case, breaking down "$\Leftrightarrow$" into "$\Rightarrow$" and "$\Leftarrow$".

Tip: rephrase mathetical jargon into English: the spans of S_1, S_2 overlap (nontrivially)

↕

when trivially means $\{0\}$.

($\Rightarrow$): $S_1 \cup S_2$ LD means (by def) that

$$\boxed{\sum_{u \in S_1} a_u \cdot u + \sum_{v \in S_2} b_v \cdot v = 0}$$

when not all a_u ; b_v are 0.

$$\underbrace{\sum_{u \in S_1} a_u \cdot u}_{\in \text{ span } S_1} = \underbrace{- \sum_{v \in S_2} b_v \cdot v}_{\in \text{ span } S_2}.$$

massage into form of span $S_1 \cap$ span S_2.

Almost done, just need to show $\sum_{u \in S_1} a_u \cdot u$ is not 0. By contradiction:

I chose LHS vector, but RHS vector is literally = to LHS, so doesn't matter

suppose fsoc that $\boxed{\sum_{u \in S_1} a_u \cdot u = 0.}$

for sake of contradiction

or

towards contradiction

Tip: pattern match! This pattern is "linear combination equals 0":

Immediately this should ring the bell of "linear independence"

Recall S_1 is LI. meaning that $\forall$ $a_u = 0$ $\forall u \in S_1$.

by def.

But then, $\sum_{v \in S_2} b_v \cdot v = 0$, and again by LI of S_2, $\Rightarrow$ $b_v = 0$ $\forall v \in S_2$.

But then we have a contradiction w/

Summary:
- use definitions of facts we are given (in this case, "LD").
- try to massage what we know into form of what we want to prove
- "follow your nose". (comes w/ experience).

(E.g. always have in mind proof techniques like contradiction?)

($\Leftarrow$): we have $\text{span } S_1 \cap \text{span } S_2 \neq \{0\}$. WTS $S_1 \cup S_2$ is LD.

$$0 \neq v = \sum_{u \in S_1} a_u \cdot u = \sum_{v \in S_2} b_v \cdot v$$

can't all be zero (b/c otherwise linear combo would be 0)

subtract: $\sum_{u \in S_1} a_u \cdot u - \sum_{v \in S_2} b_v \cdot v = v - v = 0$.

nontrivial linear combo of vectors in $S_1 \cup S_2$ that $= 0$

which is def of linear dependence.

beware: it was important that $S_1 \cap S_2 = \emptyset$ here!

Summary:
- use definition of what we are given
- follow your nose.

Remark: the ($\Rightarrow$) direction does not need $S_1 \cap S_2 = \emptyset$, but the ($\Leftarrow$) direction does. Also, the ($\Leftarrow$) direction doesn't need S_1 and S_2 to be LI.

Tip: when doing a problem that gives you certain assumptions, chances are they are all used somewhere! So be very conscious of when you need them, and when you don't.

Exercise for you!

(so e.g. try to come up w/ counterexample for the claim ($\Leftarrow$) if $S_1 \cap S_2 \neq \emptyset$).

---

when asked to find dimension, really you are asked to find basis.

§1.6. #30. Dimension of $W_1 + W_2$ and $W_1 \cap W_2$. i.e. want to find a basis of $W_1 + W_2$ or $W_1 \cap W_2$.

Procedure: - start finding a spanning set of $W_1 + W_2$ or $W_1 \cap W_2$.

(vectors inside $W_1 + W_2$ that span $W_1 + W_2$)

- remove vectors until linearly independent. (see pf on page. 22 of these notes regarding turning spanning set into basis)

7/7/22 addition

W_1+W_2:

W_1 is spanned by $\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}$

W_2 spanned by $\begin{pmatrix}0&1\\-1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}$ $\Rightarrow$ W_1+W_2 spanned by union of (5 vectors).

B/c $\begin{pmatrix}0&1\\-1&0\end{pmatrix} = \begin{pmatrix}0&1\\0&0\end{pmatrix} + (-1)\begin{pmatrix}0&0\\1&0\end{pmatrix}$ can remove that.

we have left $\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$. Remains to check LI.

(Suppose $a_1\begin{pmatrix}1&0\\0&1\end{pmatrix} + \dots + a_4\begin{pmatrix}0&0\\0&1\end{pmatrix} = \begin{pmatrix}0&0\\0&0\end{pmatrix}$

$\Rightarrow \begin{pmatrix}a_1&a_2\\a_3&a_1+a_4\end{pmatrix} = \begin{pmatrix}0&0\\0&0\end{pmatrix} \Rightarrow$ $a_1=0$ $a_2=0$ $a_3=0$ $a_1+a_4=0 \Rightarrow a_4=0$. All $a_i=0 \Rightarrow$ LI.)

For $W_1 \cap W_2$, prove it $= \left\{\begin{pmatrix}0&a\\-a&0\end{pmatrix}\right\}$.

7/5/22

Discussion Agenda:

- a theorem about LI sets being smaller than spanning sets (using the "replacement lemma), and using this to prove all bases have the same size.

"Fundamental theorem of vector spaces" or

key idea of is "replacement method"

Outline

- we start w/ a LI set L $\{u_1, \dots u_m\}$ and a spanning set S $\{v_1 \cdots v_n\}$.

Description of proof. "the LI set infiltrates the spanning set".

- Idea is to add one vector of L to S, i.e we consider $\{u_1, v_1, \dots, v_n\}$ which we know must then be LD (and spanning)

- from this we will be able to remove/displace one vector (from the original S) to get back to a spanning set of size n to get something like $\{u_1, v_1, v_2, v_4, v_5, \dots v_n\}$.

v_3 got removed / displaced.

The infiltration will slowly move all the u vectors inside of the original S, while keeping it spanning and size n.

Eventually, we will have infiltrated the entire set $\{u_1, \dots u_m\}$ into the original set S, while keeping it spanning and size n.

That will mean that $\underbrace{\{\underbrace{u_1, \dots u_m}_{\text{size } m}, \text{some } v\text{'s}\}}_{\text{size } n} \Rightarrow m \le n$.

---

initialization phrase?

Now details: let L be arbitrary LI set $\{u_1, \dots u_m\}$ (L size m)

S be arbitrary spanning set $\{v_1 \dots v_n\}$ (S size n).

Step 1

- Idea is to add one vector of L to S, i.e we consider $u_1, \underbrace{v_1, \dots, v_n}$ which we know must then be LD (and spanning)

Details: spanning obvious b/c $\{v_1 \dots v_n\} = S$ is spanning.

what about LD: well $u_1 = \sum_{i=1}^{n} a_i v_i$. (b/c $\{v_1 \dots v_n\}$ is spanning.)

~~we know $u_1 \neq 0$ (b/c $\{u_1 \dots u_m\}$ is LI) and so not all $a_i = 0$ (b/c otherwise $u_1 = \sum_{i=1}^{n} 0 \cdot v_i = 0$ ✗.)~~

true, but not necessary.

Thus, $u_1 - \sum_{i=1}^{n} a_i v_i = 0$

and b/c the coefficient is $1 \neq 0$, so this is a nontrivial linear combo of elements of $u_1, v_1, \dots v_n$.

⇓

we have that $u_1, v_1 \dots v_n$ is a LD list.

Step 2

- from this we will be able to remove/displace one vector (from the original S) to get back to a spanning set of size n

Details: $u_1, v_1, \dots, v_n$. (size $n+1$) spanning.

we said above this was LD which means we have nontrivial linear combo

$$a_1 u_1 + b_1 v_1 + \dots + b_n v_n = 0$$

non trivial means some $\underline{a_1, b_i \neq 0}$.

So $\underline{\text{let } b_k \text{ be last non zero coefficient.}}$ "cleverest part of proof"

$$\underbrace{a_1 u_1 + b_1 v_1 + \dots}_{\text{(moving ↑ to RHS, and dividing by } b_k)} + b_k v_k + 0 = 0$$

and so rearranging, and using $b_k \neq 0$ we get

$$\boxed{v_k = -\frac{1}{b_k}\left(a_1 u_1 + b_1 v_1 + \dots + b_{k-1} v_{k-1}\right)}$$

Thus, $\{u_1, v_1, \dots, v_{k-1}, v_{k+1}, \dots v_n\}$ is size n (b/c removed v_k)

and still $\underline{\text{spanning}}$! B/c of

So we have accomplished the infiltration of u_1. Now, u_2.

Step 1 tells us that $\boxed{u_1, u_2, v_1 \dots v_{k-1}, v_{k+1} \dots v_n}$

is a size $n+1$, spanning, LD.

Step 2:

repeat of before w/ minor change.

Rename $\underline{v_1 \dots v_{k-1}, v_{k+1} \dots v_n}$ to $\underline{w_1 \dots w_{n'}}$.

we said above this $u_1, u_2,$ was LD list, which means we have nontrivial linear combo

$$a_1 u_1 + a_2 u_2 + b_1 w_1 + \cdots + b_{n'} w_{n'} = 0$$

non trivial means some $a_1, a_2, b_i \neq 0$.

so let b_k be last non zero coefficient. "cleverest part of proof"

Alert: does that even exist??! In other words, in order to have b_k be the last, there needs to exist non zero b coefficients.

It is true that I can remove any vector in a LD spanning and keep it spanning. Why was I so picky about "b_k being last nonzero"?

B/c, by doing that, I ensured we were removing a w vector, not a u vector!

In order to remove a w vector, not u vector, I need some b coefficient to be nonzero.

How do I guarantee this? This is when LI of $\{u_1, \ldots u_m\}$ is used.

Suppose fsoc that all b coefficients were 0.

Then, we would have $a_1 u_1 + a_2 u_2 = 0$.

But by LI of $\{u_1, \ldots u_m\}$, $\Rightarrow$ all $a_i = 0$!

Contradicting above that "some $a_1, a_2, b_i \neq 0$" ⨳.

↳ this means that some b coefficient must have been $\neq 0$ and so we can do the exact same thing as before, and remove a v/w vector.

↳ successful infiltration of $\underbrace{\{\underline{u_1, u_2}, \text{ some } v\text{'s from original } S\}}_{\text{spanning, size } n.}$

Desired set D at stage 2, called D_2

Finally, repeat.

Step 1 shows that adding a u vector to D_k produces a size $n+1$, spanning set, Linearly dependent LD.

Step 2 shows that I can remove a v vector not a u vector! (i.e. a vector from the original S) to get D_{k+1}.

size n, spanning, and has 1 more u vector than D_k. (in fact $k+1$ many u vectors).

By doing this, at stage m, I will have produced $\underbrace{D_m}_{\text{size } n}$ which has all m u-vectors from the original L

$$\Rightarrow \quad m \leq n$$

Finally
Let's say we have 2 bases B_1, B_2.

B_1 LI, B_2 spanning $\Rightarrow$ $|B_1| \leq |B_2|$.

B_1 spanning, B_2 LI $\Rightarrow$ $|B_1| \geq |B_2|$.

$$\Downarrow$$

$$|B_1| = |B_2|.$$

This was my attempt at explaining Axler's proof in "Linear Algebra Done Right", pg. 34 (according to the pg. # in the book itself).

Thursday (and question 6 on pg. 37 of Axler)

<u>key idea</u>

- given a basis $\{v_1 \ldots v_n\}$ of some V,
  - scaling one vector preserves basis $\{cv_1, \ldots, v_n\}$ basis
  - adding one vector $\{v_1 + v_2, v_2 \ldots v_n\}$ is basis.

$$\downarrow$$

I will use this to prove rank nullity on Thursday!

7/7/22

Today:
- review the basis theorem I discussed last time
- discuss some manipulations of bases, + context of matrices
- rank-nullity theorem.

Daniel's gripe: in math classes, often you are given the statement of some problem / theorem and then you are asked to prove it.

Of course this is important skill, but I think coming up w/ the correct statement to prove in the first place is very important also, and unfortunately not taught/ at all. emphasized.

First review: (slogan)

not sets b/c sets forbid reoccurring elements

Statement: any LI list $u_1 \cdots u_m$ and any spanning list $v_1 \cdots v_n$

I claim $m \le n$.

Proof slogan: One word: **Infiltration** (of u-vectors into v-vectors)

Step 1: add one u-vector to spanning list of v-vectors to get length $n+1$ list.

Spanning $\Rightarrow$ this $n+1$ length list is LD

(slogan: add one vector to spanning list produces LD list)

Forbids something like this:

20 LI u-vectors and 4 spanning

$\{u_1 \cdots u_{20}\}$ $\{v_1 \cdots v_4\}$.

I can infiltrate u-vectors into v-vectors until I have

$\{u_5, \ldots u_{20}\}$ $\{u_1 \cdots u_4\}$

still spanning (as we saw in the course of the proof)

But adding one more u-vector

$\rightsquigarrow$ $\{u_1 \cdots u_5\}$. And **Step 1** tells us must be LD.

contradiction!

So this is exactly what "goes wrong" if we somehow had longer list of LI vectors than spanning list.

Step 2: given n+1 length spanning list w/ some u-vectors in beginning, and v-vectors at end:

Slogan: LI of u-vectors ⇒ can remove v-vector and preserve spanning

$u_1 \dots u_k$, some v's

want to add in a u-vector and remove a v-vector

then can remove one v-vector and preserve spanning.

Key! B/c in any spanning LD list, I know ∃ some vector I can remove and still preserve spanning.

The key is that $u_1 \dots u_k$ LI ⇒ can be taken to be a v-vector.

continue until all u-vectors have displaced / infiltrated the spanning list ⇝ $m \leq n$.

Most important corollary: all bases have same size (~~one~~ two liner from this theorem).

Bases: Last time student asked me about problem 6 in Axler's "LADR" pg. 37.

some VS

Suppose $v_1 \dots v_4$ is basis of V.

WTS $v_1 - v_2$, $v_2 - v_3$, $v_3 - v_4$, v_4 is basis.

(e.g. $(1,0,0,0)$ ⋮ $(0,0,0,1)$ in $V = \mathbb{R}^4$)

Theorem: ("Row operation theorem) Start w/ basis $v_1 \cdots v_n$

(a) (shuffling) • any rearrangement of $v_1 \cdots v_n$ is also basis

(b) (scaling) • can scale any vector by nonzero scalar and remain basis

(c) (adding) • $v_1 + v_2 , v_2 , v_3 . . . v_n$ also basis .

update 7/26/22: Midterm Q3

- for $\beta_a = \{av_1 \cdots av_n\}$, apply n times.
- for $\beta_{+v_1} = \{v_1+v_1 \cdots v_n+v_1\}$, apply n-1 times to get $\{v_1, v_2+v_1, \cdots, v_n+v_1\}$ is basis. Then apply to get $\{2v_1, v_2+v_1, \cdots + v_n+v_1\}$ is basis.

Pf: (a) obvious / trivial

(b) To check basis $\nearrow$ span, $\searrow$ LI

Span: let $v \in V$. B/c $v_1 \cdots v_n$ spans,

$$v = a_1 v_1 + \cdots + a_n v_n .$$

$$= \frac{a_1}{c} \cdot (cv_1) + \cdots + a_n v_n . \quad \Rightarrow \quad v \in \text{span}\{cv_1, v_2, \cdots v_n\}.$$

LI: Start w/ $a_1 (cv_1) + \cdots + a_n v_n = 0$ (WTS all $a_i = 0$).

well $(a_1 c) \cdot v_1 + \cdots + a_n v_n = 0$, by LI of $v_1 \cdots v_n$,

$$\Rightarrow \begin{array}{l} a_1 c = 0 \\ a_2 = 0 \\ \vdots \\ a_n = 0 \end{array} \quad \Rightarrow \quad \text{all } a_i = 0 \quad (\text{b/c divide by } c).$$

Corollary: above only showed could scale v_1.

But (a) and (b) together $\rightsquigarrow$ can scale anything.

(c): **spanning:** $v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$

$$= a_1 (v_1+v_2) + (a_2 - a_1) v_2 + \cdots + a_n v_n \quad \Rightarrow \quad v \in \text{span}\{v_1+v_2, v_2, \vdots, v_n\}$$

LI: if $a_1(v_1+v_2) + a_2 v_2 + \cdots + a_n v_n = 0$

then LI $v_1 \cdots v_n \Rightarrow$

$$\begin{array}{l} a_1 = 0 \\ a_1 + a_2 = 0 \\ a_3 = 0 \\ \vdots \\ a_n = 0 \end{array} \quad \Rightarrow \quad \text{all } a_i = 0 .$$

corollary: (a) (b) (c) together mean can add any scaling of any v_i to v_j and keep basis.

Math tip: if want to prove complicated thm like ↑ break it down into simplest pieces, so that all pieces together $\rightsquigarrow$ desired theorem.

At the very least, my method is using easier notation than v_i, v_j business (yucky!).

Back to Axler problem:

$$\underset{\text{basis}}{v_1 \cdots v_4} \xrightarrow[\text{thm}]{} \underset{\text{basis}}{v_1 - v_2, v_2, v_3, v_4} \xrightarrow[\text{thm}]{} \underset{\text{basis}}{v_1 - v_2, v_2 - v_3, v_3, v_4}$$

$\rightsquigarrow$ etc. ✓.

(imagine doing the problem from def LI + span and you will appreciate how much easier my thm made it.)

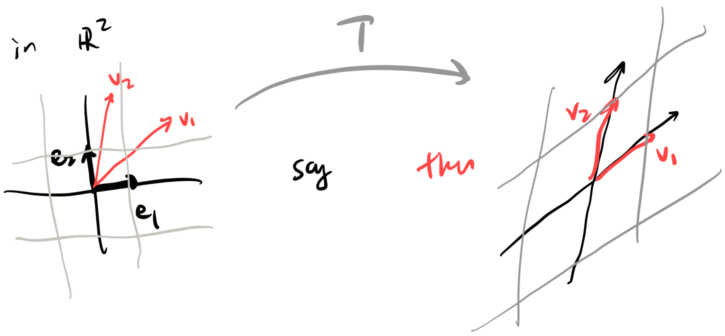
Interpretations w/ matrices: Let's work in $\mathbb{R}^m$. → has standard bas $(1,0,0,\dots 0) \dots (0,0,\dots,1)$ suppose $v_1 \cdots v_n$ is basis of $\mathbb{R}^m$. By "bases have same size" $m = n$.

I can write basis list $v_1 \cdots v_n$ as a $n \times n$-matrix

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ \vdots \\ n \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ n \end{pmatrix} = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$$

This encodes the linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ mapping standard vector $e_i \longmapsto v_i$.

$e_i = (0, 0, \dots, 1, 0 \dots 0)$ ↑ ith spot



(highly recommend 3b1b youtube video on visualizing linear transformations)

↗ WATCH

And it's $\boxed{\text{invertible} \Leftrightarrow v_1 \cdots v_n \text{ basis.}}$

left as exercise for you.

mentioned above.

The operations of scaling v_i, adding $v_i + cv_j$ is exactly column operations on the matrix $\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$

Can encode these by linear transformations / matrices.

$$\underbrace{\begin{matrix} v_1 \\ \vdots \\ v_n \end{matrix} \mapsto \begin{matrix} v_1 + v_2 \\ \vdots \\ v_n \end{matrix}}_{T}$$

$$\begin{pmatrix} 1 & 0 & & & \\ 1 & 1 & & 0 & \\ 0 & 0 & 1 & & \\ 0 & 0 & & \ddots & \\ \boxed{1 \; 0} & 0 & 0 & & \end{pmatrix}$$

(column operations matrix

↓ corresponds to

1 · first element of basis

+ 1 · 2nd element of basis.

$\boxed{\text{And theorem above is restated as this transformation is invertible}}$

- $v_1 + v_2, \cdots, v_n$ remains basis

So saying the same thing in 3 languages.:

- T is invertible

- $\begin{pmatrix} 1 & 0 & & \\ 1 & 1 & 1 & 0 \\ 0 & 0 & & \ddots \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is invertible / det $\neq 0$.

OH 7/7/22

1.6 #31 (b) Claim: suppose $u_1 \cdots u_m$ is basis of W_1, $v_1 \cdots v_n$ is basis of W_2. Then $\boxed{\begin{matrix} u_1 \cdots u_m \\ v_1 \cdots v_n \end{matrix}}$ spans $W_1 + W_2$.

⇓

basis of $W_1 + W_2$ has size $\leq m+n$.

If v is arbitrary in $W_1 + W_2$, by def $v := \overset{\in W_1}{w_1} + \overset{\in W_2}{w_2}$

$w_1 \in \text{span}\{u_1 \cdots u_m\}$ and similarly $w_2 \in \text{span}\{v_1 \cdots v_n\}$.

$\Rightarrow v := w_1 + w_2 \in \text{span}\{u_1 \cdots u_m, v_1 \cdots v_n\}$.

Explicit example: (1.6 #30 discussed above).

W_1 is spanned by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

W_2 spanned by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow$ $W_1 + W_2$ spanned by union of (5 vectors), but above we saw $\dim(W_1 + W_2) = 4$.

(HWK1)

1-2 #17 $V = \{(a_1, a_2) : a_1, a_2 \in F\}$. Elements addition componentwise.

(Proof contradiction).

assume V is VS

$c(a_1, a_2) := (a_1, 0) \quad \forall\, c \in F$.

So taking $c = 0$, $a_1 = 1$, $a_2 = 0$ we get $0 \cdot (1, 0) := (1, 0)$.

But in VS we know that $0 \cdot v = \vec{0}$ and $(1,0)$ is not $\vec{0}$ in V.

7/12/22 **Agenda:**

- Rank-nullity theorem
- VS structure of $\mathcal{L}(V,W)$ through lens of duality.
- change of basis, determinant?

Daniel's gripe: in math$\overset{\text{classes}}{\vee}$, often you are given the statement of some problem / theorem and then you are asked to prove it.

of course this is important skill, but I think coming up w/ the correct statement to prove in the first place is very important also, and unfortunately not taught/ at all. emphasized.

Recall from last discussion

Theorem: ("Row operation theorem) Start w/ basis $v_1 \cdots v_n$

(a) • (shuffling) any rearrangement of $v_1 \cdots v_n$ is also basis

(b) • (scaling) can scale any vector by nonzero scalar and remain basis

(c) • (adding) $v_1 + v_2, v_2, v_3 \ldots v_n$ also basis.

⇝ Corollary: given basis $\underline{v_1, \cdots, v_n}$, can add any scalar multiple of v_i to v_j and remain basis.

"Bases are quite tamper resistant".

Rank-nullity theorem: start by thinking about linear maps $T: V \to W$. which maps a basis $v_1 \cdots v_n$ of V to $T(v_1) \ldots T(v_n)$ in W.

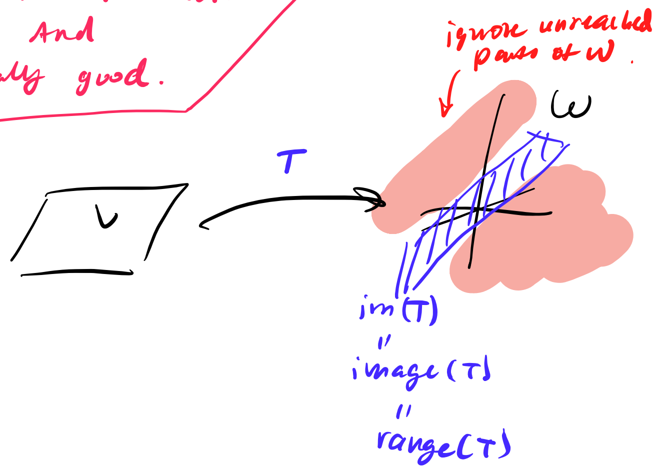
Driving question: We want to see "how far" $T(v_1) \cdots T(v_n)$ is from being basis of W

Natural Question b/c if it is basis $\Leftrightarrow$ T is invertible! And invertible maps are really good.

- spanning, LI.

↓

$T(v_1) \cdots T(v_n)$ may not span W but when just dealing w/ T, we can ignore the "unreached" parts of W b/c even if we remove them, T remains unphased.



So spanning "addressed" by restricting $T: V \to \underbrace{\text{im}(T)}_{W'}$.

LI? Given $T(v_1) \cdots T(v_n)$ $\underbrace{\text{span } \overset{\overset{W'}{\shortparallel}}{\text{im}(T)}}_{\text{exercise!}}$, we know by removal thm from (last week?) that can remove some vectors to get a basis (LI subset). By <u>relabelling</u> wrt say after removal we are left w/ basis $\underbrace{T(v_1) \cdots T(V_m)}_{\text{basis of im}(T).}$ $(m \leq n)$.

what happens to $T(v_j)$ for $j > m$?

$$\boxed{T(v_1) \quad \ldots \quad T(v_m)} \; \ldots \; T(v_n)$$

✓ extras :(

Moral: linear dependencies always lead to 0, hence the natural appearance of null(T).

well, we know that $T(v_j) \in \text{span}\{T(v_1) \cdots T(v_m)\}$.

$$T(v_j) = \sum_{i=1}^{m} a_i \, T(v_i).$$

So that means $T\left(v_j - \sum_{i=1}^{m} a_i v_i\right) = 0$. (rearranging). + linearity of T.

But from "basis tamper resistant thm",

$$v_1 \cdots v_m, \; \ldots, \; v_j - \sum_{i=1}^{m} a_i v_i, \; \ldots \; v_n$$ remains a basis!

In fact, for all $j > m$,

coefficient depending on $(m+1)$ and index i.

$$\underbrace{v_1 \cdots v_m}, \; \underbrace{V_{m+1} - \sum_{i=1}^{m} a_{(m+1),i} \, v_i}_{V'_{m+1}}, \; \ldots, \; \underbrace{V_n - \sum_{i=1}^{m} a_{(n),i} \, v_i}_{V'_n}$$

remains a basis!

where $\boxed{T(V'_{m+1}) = 0.}$

In other words, we have basis $v_1 \cdots v_n$, and we have "massaged" it into basis $T(v_1) \cdots T(v_m)$ of $\text{im}(T) = W'$, and LI set $\underbrace{V'_{m+1} \cdots V'_n}_{\in \ker(T) = \text{null}(T).}$ (kernel)

Finally, one can show $V'_{m+1} \cdots V'_n$ spans $\text{null}(T) = \ker(T)$

Suppose

Exercise?

Suppose $x \in \text{null}(T)$. Then $T(x) = 0$.

$V_1 \cdots V_m, V'_{m+1}, \ldots V'_n$ basis of V $\Rightarrow$ $x = \sum_{1}^{m} a_i v_i + \sum_{m+1}^{n} b_i V'_i$.

Applying T, $0 = T(x) = \sum_{1}^{m} a_i \, T(v_i) + 0$,

But $T(v_1) \cdots T(v_m)$ LI in $\text{im}(T)$ $\Rightarrow$ all $a_1 = \cdots = a_m = 0$. $\Rightarrow$ $x \in \text{span}\{V'_{m+1}, \ldots, V'_n\}$ ☺

"Rank Nullity thm arises from $\overset{\text{thinking about}}{\vee}$ how a basis is changed by a linear map T."

[- clever / fun / cute part: relationships among $T(v_1) \cdots T(v_n)$ ↝ 0 in W-space ↝ renamed basis in V-space.

Given now that we know the statement of R-N thm, the "quicker proof" is the one (in book?):

- start w/ basis of null(T). $u_1 \cdots u_m$. — this step requires us to know that null(T) is special, but imo that is not clear until after above proof / computation.

- extend to basis of V. technical maneuver...

  "building up method" ($\overset{\text{LI}}{\vee}$)

  - if $v_1 \cdots v_k$ does not span V, then ∃ some vector, which we call $v_{k+1} \in V$ not in span, add it to collection.
  - continue until reach $v_1 \cdots v_n$, when $n = \dim(V)$.
  - ↓ to show $v_1 \cdots v_n$ is basis. suffices to show that $\boxed{v_1 \cdots v_n}$ is LI.

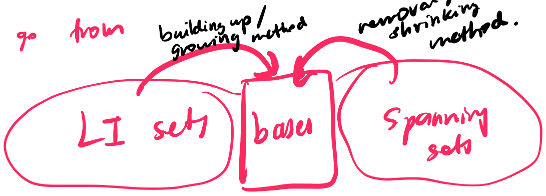
    (hint: if $\overset{v_1 \cdots v_n}{\vee}$ not spanning, ∃ another LI vector, ←
    ↝ ✗ w/ "all LI sets ≤ size of basis")
  - to show $v_1 \cdots v_n$ is LI, suppose f.s.o.c it is $\overset{\text{instead}}{\vee}$ LD ↝ $v_j \in \operatorname{span}\{v_1 \cdots v_{j-1}\}$.

    but by (construction), any subsequent vector v_j is not in span of previous.

compare to 1-2 weeks I showed you how to go from spanning set to basis. so

w/ ⟶

we now have methods to go from



↝ $u_1 \cdots u_m, v_1 \cdots v_{n-m}$ is basis of V.

- show $T(v_1) \cdots T(v_n)$ is basis of $\operatorname{im}(T) = \operatorname{range}(T)$.

Exercise?

(pg. 63 of Axler).

2 resources I strongly recommend.

Duality sneak peek: please watch 3b1b "dot products and duality".

Determinants: see my pdf

danielrui.com/papers/desertIslandMath.pdf

for my thoughts on the matter.

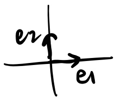
OH 7/12/22

§2.1 #14(a). "Prove T is injective $\Leftrightarrow$ maps any LI subset of V to a LI subset of W".

"(word "onto" in problem chosen very poorly IMO, b/c if T not spanning, $\exists$ LI subsets of W, namely nonzero vectors, not reached by T!).

OH 7/12/22

working in $\mathbb{R}^2$: two bases



canonical/standard $\mathcal{E} = e_1 := \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} = e_2$

$\mathcal{B}_1 = v_1 = \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}$ in the standard basis.

w.r.t. $\mathcal{E}$, meaning

$v_1 = 0 \cdot e_1 + 1 \cdot e_2$. $\underset{\text{written as}}{=} (0,1)$.

$\mathcal{B}_2 = \begin{bmatrix}-1\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$

so $\begin{bmatrix}a\\b\end{bmatrix}$ should be interpreted as

"(a,b) w.r.t. $\mathcal{E}$"

$= a \cdot e_1 + b \cdot e_2$

we have the vector

"(1,1) w.r.t. basis $\mathcal{B}_1$"

$1 \cdot v_1 + 1 \cdot v_2 = \begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$

$= \underbrace{\text{"(1,1) w.r.t. basis } \mathcal{E}\text{"}}_{=}$.

want to find "(x,y) w.r.t. basis $\mathcal{B}_2$" $= \begin{bmatrix}1\\1\end{bmatrix}$

$\overset{=}{x} \cdot \begin{bmatrix}-1\\1\end{bmatrix} + y \begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$.

"$(1,1)$ wrt. basis $\mathcal{B}_2$" $= 1\cdot\begin{bmatrix}-1\\1\end{bmatrix} + 1\cdot\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix}$.

$=$ "$(0,2)$ wrt. $\mathcal{E}$"

Daniel's understanding of change of basis.

In a VS V, suppose 2 basis, $\mathcal{B}_1 = u_1 \cdots u_n$

$\mathcal{B}_2 = v_1 \cdots v_n$

then every vector $v \in V$ can be written as $v = \sum_{i=1}^{n} a_{v,i}\cdot u_i$

$v = \sum_{i=1}^{n} b_{v,i}\cdot v_i$.

i.e. "v wrt. B_1" $= (a_{v,1}, \dots, a_{v,n}) \in \mathbb{F}^n$

"v wrt. $\mathcal{B}_2$" $= (b_{v,1}, \dots, b_{v,n}) \in \mathbb{F}^n$.

Main Takeaway: every basis $\mathcal{B}$ of V induces a linear (isomorphism) map $T_{\mathcal{B}}: V \to \mathbb{F}^n$.

Task: given a vector $\boxed{v \text{ wrt. } \mathcal{B}_1}$ "$= (a_{v,1}, \dots, a_{v,n})$", want to find $\boxed{v \text{ wrt. } \mathcal{B}_2}$ $(b_{v,1}, \dots, b_{v,n})$. via translator map $T: \mathbb{F}^n \to \mathbb{F}^n$.

we know T must map $\begin{pmatrix}a_{v,1}\\ \vdots \\ a_{v,n}\end{pmatrix} \mapsto \begin{pmatrix}b_{v,1}\\ \vdots \\ b_{v,n}\end{pmatrix}$.

Exercise: prove T linear. $\Rightarrow$ T can be written as matrix.

Now figure out what the matrix is!

we know $\begin{pmatrix} ? \end{pmatrix}\begin{pmatrix}a_{v,1}\\ \vdots \\ a_{v,n}\end{pmatrix} = \begin{pmatrix}b_{v,1}\\ \vdots \\ b_{v,n}\end{pmatrix}$.

Fact → $\begin{pmatrix} ? \end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} =$ 1st column of matrix.

That is to say, 1st column of $(?) = \begin{pmatrix}b_{v,1}\\ \vdots \\ b_{v,n}\end{pmatrix}$ for v s.t. $\begin{pmatrix}a_{v,1}\\ \vdots \\ a_{v,n}\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$.

$v = \sum_{i=1}^{n} a_{v,i}\cdot u_i \rightsquigarrow v = u_1$.

i.e. $v = u_1$ produces "column of a's" $\begin{pmatrix}1\\0\\0\\0\\\vdots\\0\end{pmatrix}$.

1st column of matrix $(?)$ = "u_1 writ. $\mathcal{B}_2$".

similarly,

$$(?) = \underbrace{\begin{pmatrix} | & & | \\ [u_1]_{\mathcal{B}_2} & \cdots & [u_n]_{\mathcal{B}_2} \\ | & & | \end{pmatrix}}_{\text{change of base matrix.}}.$$

§2.5

Example 1 textbook:

$\mathbb{R}^2$, $\mathcal{B}_2 = \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}$ $\mathcal{B}_1 = \begin{bmatrix}2\\4\end{bmatrix}, \begin{bmatrix}3\\1\end{bmatrix}$

what is $(?)$ translates from $\mathcal{B}_1$ representation to $\mathcal{B}_2$ representation?

$$(?) = \left(\boxed{\begin{bmatrix}2\\4\end{bmatrix}_{\mathcal{B}_2}}, \begin{bmatrix}3\\1\end{bmatrix}_{\mathcal{B}_2} \right).$$

$\begin{pmatrix}a\\b\end{pmatrix}$ for

matrix of $\mathcal{B}_2$

$$\begin{bmatrix}2\\4\end{bmatrix} = \underset{3}{a}\begin{bmatrix}1\\1\end{bmatrix} + \underset{-1}{b}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{pmatrix}a\\b\end{pmatrix}$$

$$\begin{bmatrix}3\\1\end{bmatrix} = 2\begin{bmatrix}1\\1\end{bmatrix} + 1\begin{bmatrix}1\\-1\end{bmatrix}$$

translator matrix $(?) = \begin{pmatrix}3&2\\-1&1\end{pmatrix}$

Let's say we have $v = \begin{pmatrix}1\\2\end{pmatrix}$ wrt. $\mathcal{B}_1$.

what is v wrt. $\mathcal{B}_2$? Exactly $\begin{pmatrix}3&2\\-1&1\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}7\\1\end{pmatrix}$.

check by hand: $\begin{pmatrix}1\\2\end{pmatrix}$ wrt. $\mathcal{B}_1$ $= 1\cdot\begin{bmatrix}2\\4\end{bmatrix} + 2\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}8\\6\end{bmatrix}$.

$\begin{pmatrix}7\\1\end{pmatrix}$ wrt. $\mathcal{B}_2$ $= 7\cdot\begin{bmatrix}1\\1\end{bmatrix} + 1\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}8\\6\end{bmatrix}$.

$\mathcal{B}_1$, $\mathcal{B}_2$ bases of V

$$\begin{tikzcd}
& V \arrow[dl, bend right, "\text{interpret wrt. } \mathcal{B}_1 \text{ (linear isomorphism)}"'] \arrow[dr, bend left, "\text{interpret wrt. } \mathcal{B}_2 \text{ (linear isomorphism)}"] & \\
\mathbb{F}^n \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \arrow[ur, "\text{multiply on left by ``matrix'' } (v_1 \cdots v_n) \text{ where } \mathcal{B}_1 = v_1 \cdots v_n"'] \arrow[rr, dashed] & & \mathbb{F}^n \arrow[ul, "\text{multiply on left by } (w_1 \cdots w_n) \text{ where } \mathcal{B}_2 = w_1 \cdots w_n"]
\end{tikzcd}$$

$(v_1 \cdots v_n) \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \sum_1^n a_i v_i$

$(w_1 \cdots w_n) \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \sum_1^n b_i w_i$

translate representation in terms of $\mathcal{B}_1$ to representation in terms of $\mathcal{B}_2$

$M = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \longmapsto \boxed{\text{Interpret } (v_1 \cdots v_n) \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \text{ wrt. } \mathcal{B}_2}$

as matrix $= \begin{pmatrix} \boxed{\text{interp } v_1 \text{ wrt. } \mathcal{B}_2} & \cdots & \boxed{\text{interp } v_n \text{ wrt. } \mathcal{B}_2} \end{pmatrix}$.

Given bases $\mathcal{B}_1$ of V and $\mathcal{B}_2$ of W, $T: V \to W$ linear map

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, shift right, "\text{interpret w.r.t. } \mathcal{B}_1"'] & W \arrow[d, "\text{interpret wrt. } \mathcal{B}_2"] \\
\mathbb{F}^m \arrow[u, shift right] \arrow[r, "M = {[T]}_{\mathcal{B}_1 \to \mathcal{B}_2}"] & \mathbb{F}^n
\end{tikzcd}$$

linear, and linear maps between $\mathbb{F}^m \to \mathbb{F}^n$ can be written as $n \times m$-matrix $\begin{pmatrix} \boxed{M\begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}} & \cdots & \boxed{M\begin{pmatrix}0\\0\\0\\0\\0\\1\end{pmatrix}} \end{pmatrix}$

$$\in \mathcal{L}(V,V)$$

$$\begin{tikzcd}
\mathbb{F}^n \arrow[dr, bend right, "\text{translate } \mathcal{B}_1 \to \mathcal{B}_2"'] & V \arrow[l, "\text{interp wrt } \mathcal{B}_1"'] \arrow[r, "T"] \arrow[d, "\text{interp wrt. } \mathcal{B}_2"'] & V \arrow[r, "\text{interp wrt } \mathcal{B}_1"] \arrow[d, "\text{interpret w.r.t. } \mathcal{B}_2"] & \mathbb{F}^n \\
 & \mathbb{F}^n \arrow[r, "M"'] & \mathbb{F}^n \arrow[ur, "\text{translate } \mathcal{B}_2 \to \mathcal{B}_1"'] &
\end{tikzcd}$$

$$M = [T]_{\mathcal{B}_2}$$

$$\Longleftrightarrow \quad [T]_{\mathcal{B}_1} = [Id_V]^{-1}_{\mathcal{B}_1 \to \mathcal{B}_2} \, M \, [Id_V]_{\mathcal{B}_1 \to \mathcal{B}_2}$$

$$[T]_{\mathcal{B}_1} = [T]_{\mathcal{B}_1 \to \mathcal{B}_1}$$

$$(v_1 \cdots v_n)\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = (w_1 \cdots w_n)\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$\Rightarrow \text{"}\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = (w_1 \cdots w_n)^{-1}(v_1 \cdots v_n)\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}\text{"}$$

if we are in $\mathbb{R}^n$, then inverse of matrix whose columns are basis vectors $w_1 \cdots w_n$ can literally be done.

More generally, if some other basis $\mathcal{B}$, then

$$\underbrace{(v_1 \cdots v_n)\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}}_{\sum a_i v_i} = \underbrace{(w_1 \cdots w_n)\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}}_{\sum b_i w_i}$$

i.e.

$$\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \left[\sum a_i v_i\right]_{\mathcal{B}_2}$$

$$\Rightarrow \begin{pmatrix} | & & | \\ [v_1]_{\mathcal{B}} & \cdots & [v_n]_{\mathcal{B}} \\ | & & | \end{pmatrix}\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} | & & | \\ [w_1]_{\mathcal{B}} & \cdots & [w_n]_{\mathcal{B}} \\ | & & | \end{pmatrix}\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

"interpreted w.r.t. $\mathcal{B}$"

$$= \sum a_i [v_i]_{\mathcal{B}} = \left[\sum a_i v_i\right]_{\mathcal{B}}$$

and then can invert the matrix to find $\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$.

Moral: if know $\mathcal{B}_1$ and $\mathcal{B}_2$ written w.r.t. some other basis $\mathcal{B}$, then easy to find change of basis matrix.

7/14/22 **Options:**
- Midterm exam + other questions people have
- Duality ⋮ vs structure of $\mathcal{L}(V,W)$

Unpolished thoughts:
- change of bases
- Eigenvalues / Eigenvectors.

Duality: A linear transformation $\mathbb{F}^n \xrightarrow{T} \mathbb{F}^m$

$$(a_1 \cdots a_n) \mapsto \underline{(b_1 \cdots b_m)}.$$

can be broken into m many linear maps $\underset{\text{rank} \leq 1 \text{ maps}}{\underbrace{\overset{T_1 \cdots T_m}{V : \mathbb{F}^n \to \mathbb{F}}}}$

rank ≤ 1 maps
$\shortparallel$
$\dim(\operatorname{im}(T))$.

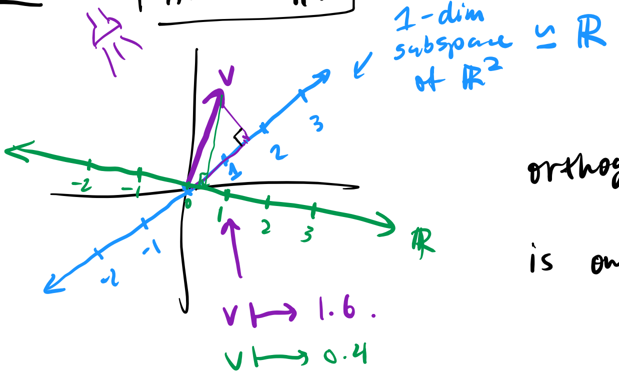
$$(a_1 \cdots a_n) \longmapsto b_1$$
$$\vdots$$
$$(a_1 \cdots a_n) \longmapsto b_m$$

Then $T = (T_1, \ldots, T_m)$ in the sense that

$$T(a_1 \cdots a_n) = (T_1(a_1 \cdots a_n), \ldots, T_m(a_1 \cdots a_n)).$$

Such maps $\mathbb{F}^n \to \mathbb{F}$ are called "linear functionals". ("basic building blocks of linear transformations").

Example: $\boxed{\mathbb{R}^2 \to \mathbb{R}}$



orthogonal projection to $\underbrace{\text{copy of } \mathbb{R} \text{ in } \mathbb{R}^2}_{\text{aka 1-dim subspace}}$ is one such map $\mathbb{R}^2 \to \mathbb{R}$.

Intuitively speaking, this is linear (to prove rigorously is actually quite challenging ⇝ one must develop a rigorous geometric picture of what orthogonal projection means).

Intuitive geometric fact: (hard to prove rigorously) orthogonal projection onto 1-dim subspaces is linear

Computationally, orthogonal projections are actually very easy.

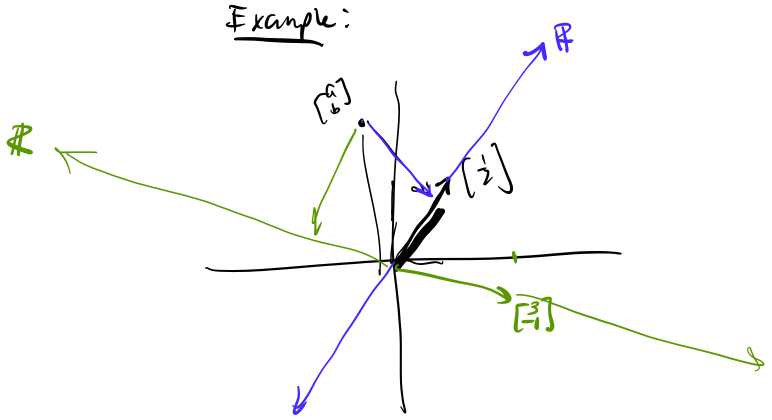
This what is known as the dot product.

So example: the orthogonal projection (up to a scaling factor) of a vector $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$ onto say $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \mathbb{R}^2$ is just given by the dot product

$$\boxed{1 \cdot a + 2 \cdot b}$$

see 3b1b video for proof .. beautiful reflection argument.

Example:



Frigyes Riesz

Are there other examples of linear functionals?

Answer: No. By the Riesz-representation theorem

all linear functionals are the result of a dot product against some vector.

In fact there is a 1-1 correspondence b/t

$$\left\{ \begin{matrix} \text{linear functionals} \\ V \to \mathbb{F} \end{matrix} \right\} \longleftrightarrow \{ \text{vectors of } V \}.$$

It is not a surprise then in view of this correspondence aka duality. that

$\mathcal{L}(V, \mathbb{F}) := \left\{ \begin{matrix} \text{linear functionals} \\ V \to \mathbb{F} \end{matrix} \right\}$ is a VS, isomorphic to V!

via the correspondence $\begin{pmatrix} a \\ b \end{pmatrix} \longleftrightarrow \left[\begin{pmatrix} x \\ y \end{pmatrix} \mapsto ax + by \right]$

tuple in say $V = \mathbb{R}^2$ — linear functional. $\mathbb{R}^2 \to \mathbb{R}$.

B/c $T: \mathbb{F}^n \to \mathbb{F}^m$ can be decomposed $T = (T_1 \cdots T_m)$

we can say $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m) = \underbrace{\mathcal{L}(\mathbb{F}^n, \mathbb{F}) \oplus \cdots \oplus \mathcal{L}(\mathbb{F}^n, \mathbb{F})}_{m \text{ copies.}}$

Thus, "m copies" VS structure of $\mathcal{L}(\mathbb{F}^n, \mathbb{F})$ (which came from duality)

should intuitively produce VS structure for $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$.

Change of bases:

suppose V, W are f.d VS, (finite dim = f.d)

Let $\underbrace{\mathcal{B}_1}_{\text{size } m} = \{v_1 \cdots v_m\}$, $\underbrace{\mathcal{B}_2}_{\text{size } n} = \{w_1 \cdots w_n\}$. be bases of V, W resp.

$$\begin{tikzcd}
V \arrow[rr, "T \text{ (linear map)}"] \arrow[d, shift right, "\text{“interpret } v\in V \text{ wrt. basis } \mathcal{B}_1\text{”}"'] & & W \arrow[d, "\text{“interpret } w\in W \text{ wrt. basis } \mathcal{B}_2\text{”}"] \\
\mathbb{F}^m \arrow[u, shift right, "\text{iso}"'] \arrow[rr, dashed, "\text{induced by } T"] & & \mathbb{F}^n
\end{tikzcd}$$

$(v_1 \cdots v_m)\begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} = a_1 v_1 + \cdots + a_m v_m$

Example: if $v = 1\cdot v_1 + 2\cdot v_2 + 0 + \cdots + (-1)\cdot v_m$

$\downarrow$

$(1, 2, 0, \ldots, 0, -1)$ m-tuple uniquely describing v.

This map is exactly some matrix multiplication $\begin{pmatrix} \text{matrix} \end{pmatrix}\begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$.

In lecture / book, this matrix is denoted $[T]_{\mathcal{B}_1}^{\mathcal{B}_2}$

I prefer $[T]_{\mathcal{B}_1 \to \mathcal{B}_2}$.

How to find matrix?

1st column of matrix

$$\underbrace{\begin{pmatrix} & \text{matrix} \end{pmatrix}}_{M} \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} = \begin{pmatrix}b_1\\ \vdots \\ b_n\end{pmatrix}$$

interpretation of $T(v_1)$

interpretation of some $v \in V$ wrt. $\mathcal{B}_1$.

What is v? Exactly

$v = 1 \cdot v_1 + 0 + \cdots 0$
$= v_1$

$$\begin{tikzcd}
v_1 \in V \arrow[r, "T"] & T(v_1) \in W \arrow[d] \\
\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} \in \mathbb{F}^n \arrow[u] & \begin{pmatrix}b_1\\ \vdots \\ b_n\end{pmatrix}
\end{tikzcd}$$

$\begin{pmatrix}b_1\\ \vdots \\ b_n\end{pmatrix}$ is exactly interpretation of $T(v_1)$ wrt. $\mathcal{B}_2$.

Moral

choosing basis allows iso $V \to \mathbb{F}^n$
$W \to \mathbb{F}^n$

$\rightsquigarrow$ induces

ability to write $T: V \to W$ as matrix.

Ie. 1st column of matrix M is interpretation of $T(v_1)$ wrt. $\mathcal{B}_2$.

Ultimately

$$M = \begin{pmatrix} \vert & & \vert \\ \boxed{\begin{matrix} T(v_1) \\ \text{wrt. } \mathcal{B}_2 \end{matrix}} & \cdots & \boxed{\begin{matrix} T(v_m) \\ \text{wrt } \mathcal{B}_2 \end{matrix}} \\ \vert & & \vert \end{pmatrix}$$

Special cases:

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "\text{iso}"', "\text{via basis } \mathcal{B}_1"] & V \arrow[d, "\text{iso}", "\text{via basis } \mathcal{B}_2"'] \\
\mathbb{F}^n \arrow[r, dashed, "M = {[T]}_{\mathcal{B}_1 \to \mathcal{B}_2}"'] & \mathbb{F}^n
\end{tikzcd}$$

induced map = some matrix multiplication

① if $T = Id_V$

$$\begin{tikzcd}
& V \arrow[dl, "\text{iso}", "\text{via basis } \mathcal{B}_1"'] \arrow[dr, "\text{iso via basis } \mathcal{B}_2"] & \\
\mathbb{F}^m \arrow[rr, dashed] & & \mathbb{F}^m
\end{tikzcd}$$

$\mathcal{B}_1 = \{v_1 \cdots v_m\}.$

$M = [Id_V]_{\mathcal{B}_1 \to \mathcal{B}_2}$

$\boxed{\text{change of basis matrix}}$

Moral : triangle diagram is change of basis matrix

$$M = \begin{pmatrix} | & & | \\ \boxed{v_1 \text{ wrt. } \mathcal{B}_2} & \cdots & \boxed{v_m \text{ wrt } \mathcal{B}_2} \\ | & & | \end{pmatrix}$$

② if $\mathcal{B}_1 = \mathcal{B}_2$

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "\text{interpret via } \mathcal{B}_2"'] & V \arrow[d, "\text{interpret via } \mathcal{B}_2"] \\
\mathbb{F}^m \arrow[r, dashed] \arrow[r, bend left=20, purple] \arrow[r, red, bend left=80] & \mathbb{F}^m
\end{tikzcd}$$

$M = [T]_{\mathcal{B}_2}$

notation from book.

$[T]_{\mathcal{B}_2}$ is the unique matrix that makes this picture / diagram commutes.

i.e. where

$\mathcal{B}_2 = \{v_1 \cdots v_m\}.$

$$M = \begin{pmatrix} | & & | \\ \boxed{T(v_1) \text{ wrt. } \mathcal{B}_2} & \cdots & \boxed{T(v_m) \text{ wrt } \mathcal{B}_2} \\ | & & | \end{pmatrix}$$

Finally putting together:

$$\begin{tikzcd}
 & V \arrow[r, "T"] \arrow[dl, "\text{via } \mathcal{B}_1"'] \arrow[d, "\text{via } \mathcal{B}_2"] & V \arrow[d, "\text{via } \mathcal{B}_2"] \arrow[dr, "\text{via } \mathcal{B}_1"] & \\
\mathbb{F}^n \arrow[r, dashed] & \mathbb{F}^m \arrow[r, dashed, "{[T]_{\mathcal{B}_2}}"'] & \mathbb{F}^m \arrow[r, dashed, "{[Id_V]_{\mathcal{B}_2 \to \mathcal{B}_1}}"] & \mathbb{F}^m
\end{tikzcd}$$

change of basis matrix $\mathcal{B}_1 \to \mathcal{B}_2$

$\shortparallel$

$[Id_V]_{\mathcal{B}_1 \to \mathcal{B}_2}$

$\shortparallel$

Q

$= [T]_{\mathcal{B}_1}.$

uniqueness forces equality here.

in textbook, $[T]_{\mathcal{B}_1} = Q [T]_{\mathcal{B}_2} Q^{-1}$.

Finally: how do I actually compute "w wrt. $\mathcal{B}_2$"?

we can do this "sort of easily" in special case.

$$\begin{tikzcd}
\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} & & \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \\
 & V \arrow[ul, mapsto, "\{v_1 \cdots v_n\} = \mathcal{B}_1"] \arrow[ur, mapsto, "\mathcal{B}_2 = \{w_1 \cdots w_n\}."'] &
\end{tikzcd}$$

given $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ and $\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

this means that

$$\underbrace{(v_1 \cdots v_n) \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}}_{a_1 v_1 + \cdots + a_n v_n} = v = \underbrace{(w_1 \cdots w_n) \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}}_{b_1 w_1 + \cdots + b_n w_n}.$$

(by definition basically).

so given $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, want to find $\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$.

$$\left(\underbrace{(M)}_{\text{find } M!} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \right).$$

Idea: "$\underbrace{(w_1 \cdots w_n)^{-1} (v_1 \cdots v_n)}_{= M = \text{change of basis matrix } \{v_1 \cdots v_n\} \to \{w_1 \cdots w_n\}.} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$"

Amazingly: this works in $\mathbb{F}^n$!

representing vectors like $v_1, \dots, v_n$, $w_1, \dots, w_n \in \mathbb{F}^n$ as column tuples of elements of $\mathbb{F}$,

$$M = \underbrace{\begin{pmatrix} | & & | \\ w_1 & \cdots & w_n \\ | & & | \end{pmatrix}^{-1}} \underbrace{\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}}$$

actually form matrices.

(pf / generalization on pg. 45 of my notes, good luck reading it).

Example: in $\mathbb{R}^2$, basis $\left\{ \overset{v_1}{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}, \overset{v_2}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} \right\} = \mathcal{B}_1$,

$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\} = \mathcal{B}_2$

Then change of basis matrix

$$\begin{tikzcd} & \mathbb{R}^2 \arrow[dl, "\text{via } \mathcal{B}_1"'] \arrow[dr, "\text{via } \mathcal{B}_2"] & \\ \mathbb{R}^2 \arrow[rr, dashed, "M"'] & & \mathbb{R}^2 \end{tikzcd}$$

$$= \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}^{-1} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{-2} \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} -1 & 1 \\ 3 & -1 \end{bmatrix} = M$$

check

$$\begin{tikzcd} M\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = & \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \\ V \in \mathbb{R}^2. \arrow[u] \arrow[ur] & \end{tikzcd}$$

want to find $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, i.e. representation of $\begin{pmatrix} 5 \\ 7 \end{pmatrix} \in \mathbb{R}^2$ wrt. $\mathcal{B}_1$.

$$a_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$$

$$\Updownarrow$$

$$\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$$

$$\Updownarrow$$

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ 7 \end{pmatrix}.$$

want to find

$$\begin{tikzcd} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} & & \begin{bmatrix} 5 \\ 7 \end{bmatrix}. \\ & \begin{bmatrix} 5 \\ 7 \end{bmatrix} \arrow[ul, "\mathcal{B}_1"] \arrow[ur] & \end{tikzcd}$$

$\mathcal{E} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ standard basis

By formula:

$$\underbrace{\begin{bmatrix} \text{change of basis from} \\ \mathcal{E} \to \mathcal{B}_1 \end{bmatrix}}_{\begin{matrix} \parallel \\ \underset{\mathcal{B}_1}{\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}^{-1}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \end{matrix}} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}.$$

OH 7/14/22

WTS T injective $\Leftrightarrow$ T maps LI sets in U $\longmapsto$ LI sets in W.

$(\Leftarrow)$: contradiction.

~~Suppose f.s.o.c T is not injective.~~

~~Then, $\exists\, v \neq 0$ st. $\boxed{T(v)=0}$~~

~~→ Then $\{v\}$ is LI in V b/c $\forall$ nonzero vectors ☺ of V, $\{☺\}$ is LI set of V.~~

~~pg 38 in textbook.~~

~~But T maps $\{v\} \longmapsto \{0\}$~~ not LI.

Suppose T is not injective.

We also assume T map LI sets $\longmapsto$ LI sets.

Goal is to reach contradiction

→ by def, $\exists\, v \neq 0$ st. $T(v) = 0$.

Fact: V is VS, $v \in V$ nonzero. Then $\{v\}$ is LI in V.

pf: have linear combo $av = 0$. But $v \neq 0 \Rightarrow a = 0$.

thus linear combo is trivial.

↳ $\{v\}$ is LI set in V.

T maps $\{v\}$ to a LI set.

But what does T actually map $\{v\}$ to?

T map $\{v\}$ to set $\{0\}$.

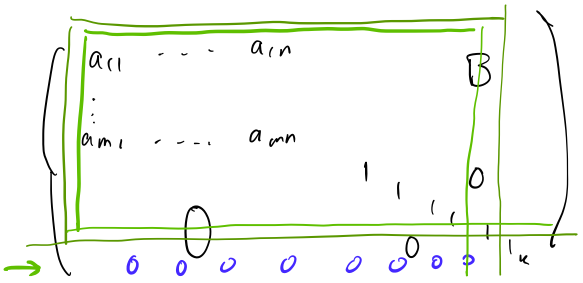
LD.

If S is set,

$$T(S) := \{\, T(x) : x \in S \,\}.$$

7/19/22 4-5 pm OH

§4.4 #5 $\begin{pmatrix} A & B \\ 0 & I \end{pmatrix}$



Idea:
cofactor expansion w.r.t. bottom row

$1 \cdot \det \boxed{\text{green}}$ + loads of zeroes.

By induction: $\det \begin{pmatrix} A & B \\ 0 & I_k \end{pmatrix} = 1 \cdot \det \begin{pmatrix} A & \overset{\boxed{\text{truncated}}}{B} \\ 0 & I_{k-1} \end{pmatrix}.$

$= 1 \cdot \det \boxed{\text{light green}}$

$\vdots$

$= 1 \cdot \det(A).$

§5.1 #11

𝔅 is list $v_1, \dots, v_n$ \mapsto $\longmapsto$

Find matrix of $\underbrace{\lambda Id_V := [v \mapsto \lambda v]}_{\text{transformation } T}$ w.r.t. 𝔅.

Fact: matrix $[T]_{\mathcal{B}}$ when 𝔅 is list $v_1 \cdots v_n$

$$= \begin{pmatrix} \boxed{Tv_1 \text{ wrt } \mathcal{B}} & \cdots & \overset{\text{jth column}}{\downarrow} & \cdots & \boxed{Tv_n \text{ wrt } \mathcal{B}} \end{pmatrix} = \begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{n1} & \cdots & m_{nn} \end{pmatrix}.$$

$$\underbrace{\begin{pmatrix} & \big| & \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}}$$ ← jth spot.

is $\underbrace{\boxed{Tv_j \text{ wrt. } \mathcal{B}.} = \begin{pmatrix} m_{1j} \\ m_{2j} \\ \vdots \\ m_{nj} \end{pmatrix}}$ = jth col of

⇓ by def of "wrt. 𝔅"

$$Tv_j = \underline{m_{ij} \cdot v_1 + \dots + m_{nj} \cdot v_n}.$$

$$= \sum_{i=1}^{n} m_{ij} \cdot v_i$$

if M diagonal, that means that if $i \neq j$, $m_{ij} = 0$.

$$\begin{pmatrix} \lambda_1 & & \overset{\text{jth column}}{\boxed{\begin{matrix} 0 \\ 0 \\ 0 \\ \lambda_j \\ 0 \\ 0 \end{matrix}}} & \\ & \lambda_2 & & \\ & & & \lambda_n \end{pmatrix}$$

§5.1 #20

$$\text{characteristic polynomial} = \underset{\uparrow}{\det(A - tI)} . = f(t) := (-1)^n t^n + \cdots + a_0$$

plug in $t=0$, $\det(A) = \_\_\_ \quad \_\_\_\_ \quad = a_0$

Fact (from class?) : A inv$^{\text{ble}}$ $\Leftrightarrow \det A \neq 0 \Leftrightarrow a_0 \neq 0$.

7/19/20 class notes review

$$\begin{array}{ccc} \langle cf \cdot g \rangle & \text{WTS} = & c\langle f, g \rangle . \\ \shortparallel & & \shortparallel \\ \int_0^1 cf(t)\, g(t)\, dt & = & c \int_0^1 f(t)\, g(t)\, dt \end{array}$$

by integral theory .

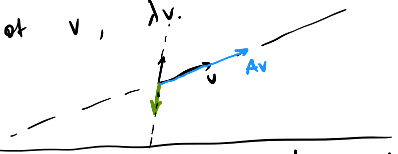
7/19/22 **Agenda**: - motivating / intuition eigenvalues (via SVD), and Fibonacci example

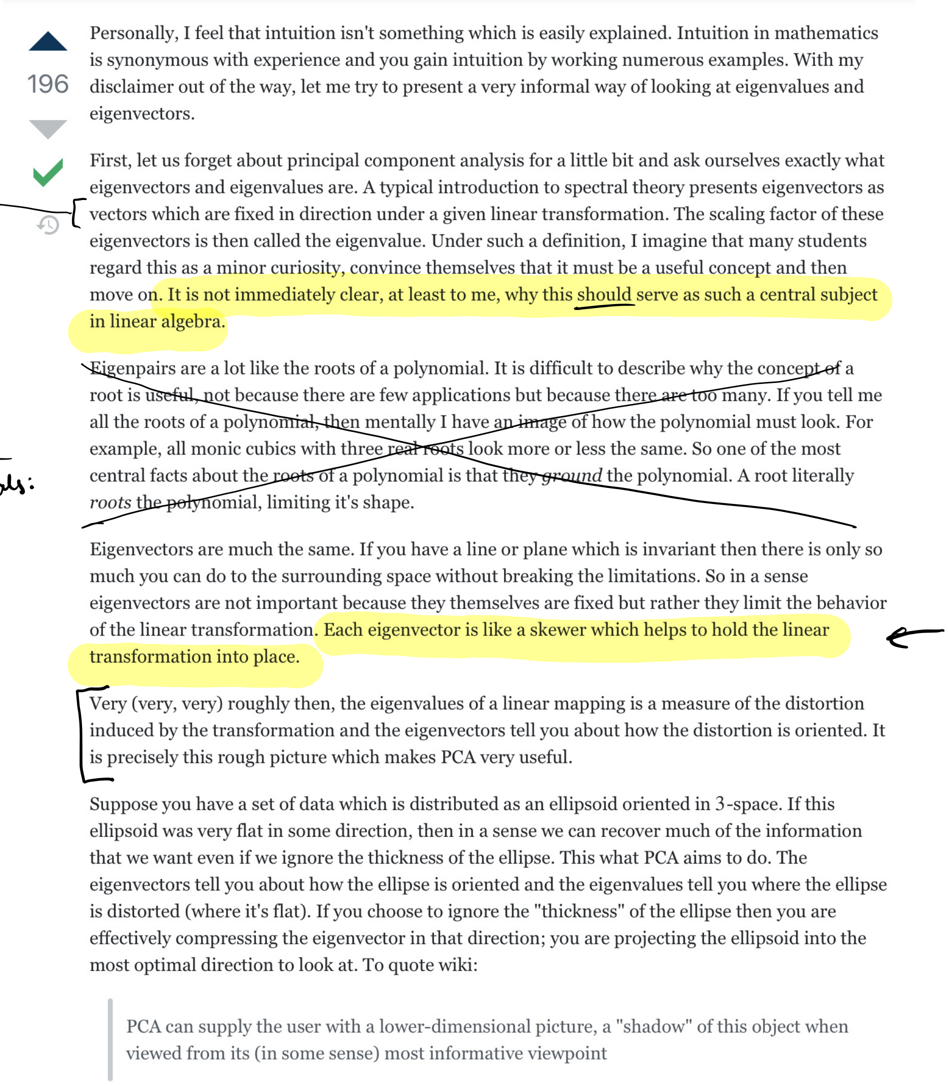
MSE "How to intuitively understand eigenvalue and eigenvector"

User @EuYu's answer:

Def: v is Evec if $\exists \lambda \in F$ st. $Av = \lambda v$. λ is Eval assoc. w/ Evec v.

A acts on v in "simplest possible way", i.e. Av is just a scaled copy of v, λv.





Personally, I feel that intuition isn't something which is easily explained. Intuition in mathematics is synonymous with experience and you gain intuition by working numerous examples. With my disclaimer out of the way, let me try to present a very informal way of looking at eigenvalues and eigenvectors.

First, let us forget about principal component analysis for a little bit and ask ourselves exactly what eigenvectors and eigenvalues are. A typical introduction to spectral theory presents eigenvectors as vectors which are fixed in direction under a given linear transformation. The scaling factor of these eigenvectors is then called the eigenvalue. Under such a definition, I imagine that many students regard this as a minor curiosity, convince themselves that it must be a useful concept and then move on. It is not immediately clear, at least to me, why this should serve as such a central subject in linear algebra.

~~Eigenpairs are a lot like the roots of a polynomial. It is difficult to describe why the concept of a root is useful, not because there are few applications but because there are too many. If you tell me all the roots of a polynomial, then mentally I have an image of how the polynomial must look. For example, all monic cubics with three real roots look more or less the same. So one of the most central facts about the roots of a polynomial is that they *ground* the polynomial. A root literally *roots* the polynomial, limiting it's shape.~~

Eigenvectors are much the same. If you have a line or plane which is invariant then there is only so much you can do to the surrounding space without breaking the limitations. So in a sense eigenvectors are not important because they themselves are fixed but rather they limit the behavior of the linear transformation. Each eigenvector is like a skewer which helps to hold the linear transformation into place.

Very (very, very) roughly then, the eigenvalues of a linear mapping is a measure of the distortion induced by the transformation and the eigenvectors tell you about how the distortion is oriented. It is precisely this rough picture which makes PCA very useful.

Suppose you have a set of data which is distributed as an ellipsoid oriented in 3-space. If this ellipsoid was very flat in some direction, then in a sense we can recover much of the information that we want even if we ignore the thickness of the ellipse. This what PCA aims to do. The eigenvectors tell you about how the ellipse is oriented and the eigenvalues tell you where the ellipse is distorted (where it's flat). If you choose to ignore the "thickness" of the ellipse then you are effectively compressing the eigenvector in that direction; you are projecting the ellipsoid into the most optimal direction to look at. To quote wiki:

> PCA can supply the user with a lower-dimensional picture, a "shadow" of this object when viewed from its (in some sense) most informative viewpoint

I give several reasons for studying Evals:

① a more "pure mathematical" reason for eigenstuffs, based on intuitive geometric picture

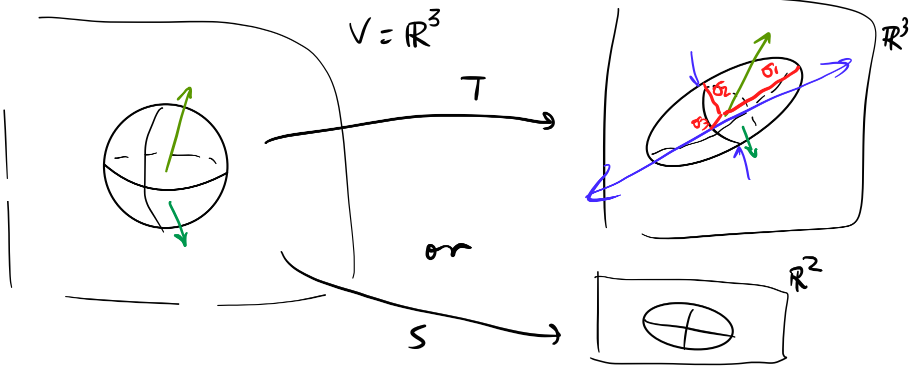
② computational perspective ("applied math" reason)

③ useful properties of Evals.

→ **SVD** singular value decomposition

Geometric observation (intuition only, turns to be rather hard to prove): linear transformations map circles to some oval, in particular, exactly an ellipse.

More generally, spheres ⟼ ellipsoids



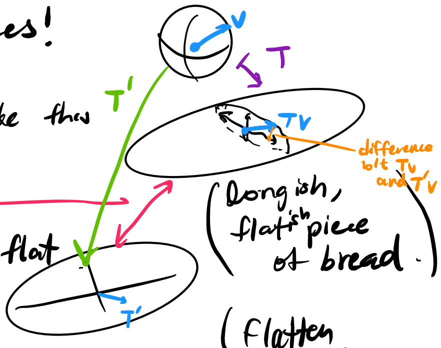
$\sigma_1, \sigma_2, \sigma_3$ lengths of axes of ellipsoid, usually arranged decreasingly, called "singular values".

Then: you see that knowing the shape of ellipsoid = image of unit sphere tells us a lot about transformation!

In particular very good visual intuition.

(Also sidenote: one very good way to compress matrices!

suppose ellipsoid is like this



very close to just flat ellipse

(long ish, flat^ish piece of bread.)

(flatten bread completely)

Intuition: for v close to unit sphere, Tv close to T'v.

But T' is simpler than T. B/c $T: \mathbb{R}^3 \to \mathbb{R}^3$
$T': \mathbb{R}^3 \to \mathbb{R}^2$.

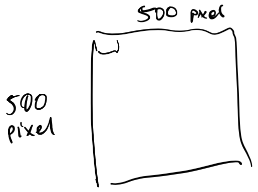
So matrix of T' is $2 \times 3 = 6$ entries

matrix of T is $3 \times 3 = 9$ entries.

So use 66% the amt of space.

Depends on ellipse being flat-ish ⟺ some σ_i are small.
singular values.

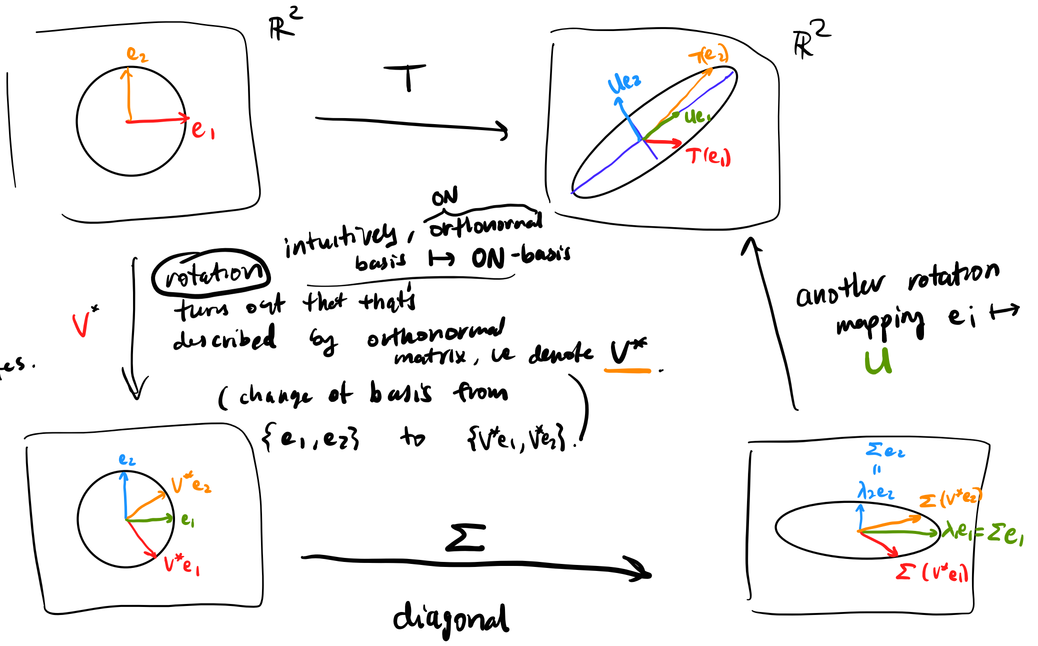
Application: data/image compression.



, maybe reduced to 20 singular values, and 20 corresponding ellipsoid axes = vectors
500x1

So how exactly does it work mathematically?

(picture from wiki)



Notice how Te_1, Te_2 not aligned w/ ellipse axes. Goal now, find good coordinates.

rotation turns out that that's described by orthonormal matrix, we denote V^*.

intuitively, $\overset{ON}{\overbrace{\text{orthonormal}}}$ basis ↦ ON-basis

(change of basis from $\{e_1, e_2\}$ to $\{V^*e_1, V^*e_2\}$.)

another rotation mapping $e_i \mapsto$ unit vector along ith axis of ellipse.

U

diagonal
(stretch along vertical/horizontal)

Decomposed T into 3 maps: rotation, then horizontal/vertical stretch, then rotation again.

Now: how to find Σ and U and V^*.

Fact about orthonormal matrices:

$$\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$$

and $v_1 \cdots v_n$ are orthonormal

orthogonal ↘ perpendicular, (mathematically, innerproduct ⟺ $\langle v_i, v_j\rangle = 0$ if $i \neq j$.)

normal = unit vector

$\langle v_i, v_i\rangle = \|v_i\|^2 = 1$

⇑

all v_i are unit vectors, and $\langle v_i, v_j\rangle = 0$ for $i \neq j$

⇓

$\langle v_i, v_j\rangle = 1_{[i=j]}$.

Algebraic property of ON-matrices

$$\underset{V^T}{\begin{pmatrix} — v_1 — \\ \vdots \\ — v_n — \end{pmatrix}} \underset{V}{\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}} = \begin{pmatrix} \langle v_1, v_1\rangle & \langle v_1, v_2\rangle & \\ \langle v_2, v_1\rangle & \ddots & \\ \vdots & & \ddots \end{pmatrix}$$

dot products

|| ⟺ v_i's are ON basis

$$\begin{pmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \\ & & 1 & \\ & 0 & & \ddots \end{pmatrix} = I$$

i.e. $\underbrace{V^TV = I}_{\text{very algebraic idea}}$ ⟺ $\underbrace{V \text{ is ON basis}}_{\text{very geometric idea}}$. , things perpendicular to each other

i.e. $V^T = V^{-1}$.

see MSE for $AB = I \Rightarrow BA = I$ ☆

Alert over ℂ, above discussion replace V^T w/ V^*.

(over field ℝ, $V^* = V^T$ so just go w/ it for now).

for now, over ℝ

so $\overset{\text{matrix}}{T} = U \Sigma V^T$, use formula!

$$T^T = (V^T)^T \Sigma^T U^T$$
$$= V \underset{\text{b/c diagonal}}{\Sigma} U^T .$$

Now
$$TT^T = U\Sigma \underset{\underset{I}{\|}}{\boxed{V^T V}} \Sigma U^T$$
$$= U \Sigma^2 U^T .$$
$$= U \Sigma^2 U^{-1}$$

OMG! This is exactly statement that TT^T is diagonalizable matrix, and Σ^2 is diagonal matrix filled w/ Evals of TT^T.

SO, $\{$singular values of $T\}$ = $\{$square root of Evals of $TT^T\}$.

and, U = Evecs of TT^T.

Similarly, V = Evecs of $T^T T$

To compute SVD of T boils down to Evals & Evecs of TT^T or $T^T T$.

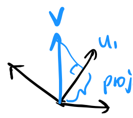
7/21/22 Discussion

<u>Agenda</u>:

- Brief overview of orthogonal things, Gram-Schmidt.
- Computational motivation for EVal decomp using Fibonacci example. (+ relation to Differential Equations).

- why ON orthonormal bases $\underline{u_1 \cdots u_n}$ good?

B/c can easily find representation of any vector $v \in V$; (using inner product) namely just $v = \sum_{i=1}^{n} \underbrace{\langle v, u_i \rangle}_{\text{geometric intuition: } \langle v, u_i \rangle \text{ is length of projection of } v \text{ down to } u_i.} \cdot u_i$.



projection of v down to u_1 is exactly "how much u_1 is in v". and orthogonality allows us to sum these over ON-basis.

Compared to what? Regular bases $\underline{v_1 \cdots v_n, \text{ say in } \mathbb{F}^n}$, how to find representation of v wrt. $v_1 \cdots v_n$? If you recall, change of basis matrix is exactly

So $$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}^{-1} \begin{pmatrix} | \\ v \\ | \end{pmatrix}$$ satisfies $v = \sum_{i=1}^{n} a_i v_i$.

all v's as column n-tuples

(Reminder, pf: have $$\begin{pmatrix} | \\ v \\ | \end{pmatrix} = a_1 \begin{pmatrix} | \\ v_1 \\ | \end{pmatrix} + a_2 \begin{pmatrix} | \\ v_2 \\ | \end{pmatrix} + \cdots + a_n \begin{pmatrix} | \\ v_n \\ | \end{pmatrix}$$
$$= \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$
by def matrix-vector mult

$$\Downarrow$$

$$\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}^{-1} \begin{pmatrix} | \\ v \\ | \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}.$$)

So for ON basis $\mathcal{B}$, representation of v wrt. $\mathcal{B}$ is as hard as inner product.

for ordinary basis $\mathcal{B}$, ―――――――――― as hard as <u>matrix inversion</u>. hard ish in general

And in fact, as you perhaps suspect, a corollary is that inverse of ON-basis matrix is easy to compute. Recall last time that inverse is exactly

$$\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}^{T} = \begin{pmatrix} — u_1 — \\ \vdots \\ — v_n — \end{pmatrix}$$

and moreover sanity check just to verify using above formula

conjugate transpose if over $\mathbb{C}$.

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix}^{-1} \begin{pmatrix} | \\ v \\ | \end{pmatrix} = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix}^{T} \begin{pmatrix} | \\ v \\ | \end{pmatrix}$$

$$= \begin{pmatrix} — & u_1 & — \\ & \vdots & \\ — & u_n & — \end{pmatrix} \begin{pmatrix} | \\ v \\ | \end{pmatrix} = \begin{pmatrix} u_1 \cdot v \\ \vdots \\ u_n \cdot v \end{pmatrix}$$

So indeed $a_i = u_i \cdot v = \langle u_i, v \rangle$ as I promised.

Summary: ON-basis / $\underbrace{\text{op-matrix}}_{\text{columns are ON-basis}}$ good b/c

related fact {
- find representation of $v \in V$ wrt. $\mathcal{B}$ is easy as $\langle \cdot, \cdot \rangle$,
- inverse is easy to find

Gram-Schmidt process

key idea:

$$\boxed{v - \boxed{\text{projection of } v \text{ down to subspace } M}} \text{ is } \underbrace{\text{orthogonal to } M.}$$

$$\| $$

Σ of projection of v down to a basis of M.

+ inductively building up ON-basis.

$M_1 = \text{span}\{v_1\}$.

v_2' is $v_2 - [\text{proj of } v_2 \text{ down to subspace } M_1]$.

$\hookrightarrow$ v_1, v_2' (normalized to unit length) is now ON-basis for $M_2 := \text{span}\{v_1, v_2\}$.

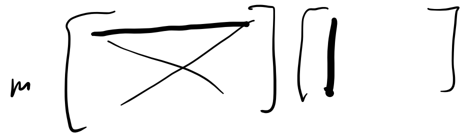
v_3' is $v_3 -$ [proj of v_3 down to subspace M_2]

$\hookrightarrow$ v_1, v_2', v_3' (normalized to unit length) is now ON-basis for $M_3 := \text{span}\{v_1, \dots v_3\}$.

⋮

Computational motivation for Evals/diagonalization.

Matrices are very much 2-dimensional objects.



and multiplying matrices is therefore quiet costly. Just finding one element requires n-multiplications and $\approx$ n-sums. In total, for $n \times n$ matrix, require $\approx n^3$ multiplications and $\approx n^3$ sums.

So, lets say now we want to track dynamics of transformation T, i.e. given x, what does orbit/trajectory $x, Tx, T^2x = T(Tx), T^3x, \dots$

Understanding T^{100000} is practically impossible for humans to do (by hand).

Idea, find "good basis" for T st. the matrix wrt $\mathcal{B}$ is diagonal.

Diagonal matrix 1-dimenal object, and raising to powers / multiplying in general is trivial!

$$\begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}^p = \begin{pmatrix} d_1^p & & 0 \\ & \ddots & \\ 0 & & d_n^p \end{pmatrix}.$$

easiest matrices to work w/

For diagonal matrices (and no others, I think in some rigorous way), powers T^k are easy to compute.

hmm, nilpotent matrices.

That's why Eval decomp is so desired in computations.

Application: Fibonacci!

$F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

Recomendation: search up cool Fibonacci # facts! (Number theory).

Goal : find closed form formula for F_n.

Clever idea, involve linear algebra!

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A}\begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix} = \begin{bmatrix} F_{n-1}+F_{n-2} \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}.$$

A "upgrades" tuple of consecutive Fib #s $\rightsquigarrow$ next tuple of consecutive Fib #s.

So, $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^1 \begin{bmatrix} \overset{F_1}{\overset{\shortparallel}{1}} \\ \underset{F_0}{\underset{\shortparallel}{0}} \end{bmatrix} = \begin{bmatrix} F_2 \\ F_1 \end{bmatrix}$.

$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} F_2 \\ F_1 \end{bmatrix} = \begin{bmatrix} F_3 \\ F_2 \end{bmatrix}$.

So, F_n = 2nd coordinate in $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

So to find formula for A^n, use above mentioned strategy: Eval decomp.

Evals of A : roots of $\det(A-\lambda I)$: $=0$

$$\det\begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = 0.$$

$$(1-\lambda)(-\lambda) - 1\cdot 1 = 0$$

roots $\lambda^2-\lambda-1=0$. quadratic formula: $\lambda = \underbrace{\overset{\approx 1.618}{\frac{1+\sqrt{5}}{2}}}_{\varphi}$, $\underbrace{\overset{\approx -0.618}{\frac{1-\sqrt{5}}{2}}}_{\bar{\varphi}}$

Golden ratio!

Evecs of A :

- $\begin{bmatrix} 1-\varphi & 1 \\ 1 & -\varphi \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

must check all such vectors are in nullspace.

Both directions necessary

if in nullspace, must be of this form

$x_1 - x_2\varphi = 0$

$\Rightarrow x_1 = x_2\varphi$. So Eigenspace assoc φ is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \boxed{\begin{bmatrix} x_2\varphi \\ x_2 \end{bmatrix}}$

$= x_2\begin{bmatrix} \varphi \\ 1 \end{bmatrix}$

i.e. $\operatorname{span}\left\{\begin{bmatrix} \varphi \\ 1 \end{bmatrix}\right\}$.

- $\begin{bmatrix} 1-\bar{\varphi} & 1 \\ 1 & -\bar{\varphi} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 - x_2\bar{\varphi} = 0$

$x_1 = x_2\bar{\varphi}$ $\Rightarrow$ Espace assoc $\bar{\varphi}$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2\bar{\varphi} \\ x_2 \end{bmatrix}$

$= \operatorname{span}\left\{\begin{bmatrix} \bar{\varphi} \\ 1 \end{bmatrix}\right\}$.

By diagonalization formula:

$$A = PDP^{-1}$$

$$= \begin{bmatrix}\varphi & \bar{\varphi}\\ 1 & 1\end{bmatrix}\begin{bmatrix}\varphi & 0\\ 0 & \bar{\varphi}\end{bmatrix}\begin{bmatrix}\varphi & \bar{\varphi}\\ 1 & 1\end{bmatrix}^{-1} = \begin{bmatrix}\varphi & \bar{\varphi}\\ 1 & 1\end{bmatrix}\begin{bmatrix}\varphi & 0\\ 0 & \bar{\varphi}\end{bmatrix}\left(\frac{1}{\sqrt{5}}\begin{bmatrix}1 & -\bar{\varphi}\\ -1 & \varphi\end{bmatrix}\right).$$

So $$A^n = (PDP^{-1})^n$$

$$= \underbrace{PD\cancel{P^{-1}P}DP^{-1}\ldots}_{n \text{ times}} = PD^nP^{-1}$$

So $$A^n = \begin{bmatrix}\varphi & \bar{\varphi}\\ 1 & 1\end{bmatrix}\begin{bmatrix}\varphi^n & 0\\ 0 & \bar{\varphi}^n\end{bmatrix}\left(\frac{1}{\sqrt{5}}\begin{bmatrix}1 & -\bar{\varphi}\\ -1 & \varphi\end{bmatrix}\right)$$

$$\begin{bmatrix}F_{n+1}\\ F_n\end{bmatrix} = \downarrow \qquad \begin{bmatrix}1\\ 0\end{bmatrix}$$

$$= \frac{1}{\sqrt{5}}\begin{bmatrix}\varphi^{n+1} - \bar{\varphi}^{n+1}\\ \varphi^n - \bar{\varphi}^n\end{bmatrix}.$$

Conclusion. $$F_n = \frac{1}{\sqrt{5}}\left(\varphi^n - \bar{\varphi}^n\right)$$

$$= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

Amazing! Found a formula! Most shocking thing is that I know w/ 100% certainty that this always produces an integer for $n \in \mathbb{N}$.

To leave off, can raise e to power of matrix:

Recall $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^A := \sum_{n=0}^{\infty} \frac{A^n}{n!} = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{PD^nP^{-1}}{n!} = \lim_{N\to\infty}\left(P\left(\sum_{n=0}^{N}\frac{D^n}{n!}\right)P^{-1}\right)$$

$$= P \begin{pmatrix} \sum_{n=0}^{\infty} \frac{d_1^n}{n!} & & \\ & \ddots & \\ & & \sum_{n=0}^{\infty} \frac{d_m^n}{n!} \end{pmatrix} P^{-1} = P \begin{bmatrix} e^{d_1} & & \\ & \ddots & \\ & & e^{d_m} \end{bmatrix} P^{-1}.$$

Why useful? Turns out this is exactly how to find solution to system of ODE's!

$$\underline{\vec{x}' = A\vec{x}} \qquad \left(\begin{aligned} \frac{dx}{dt} &= a_1 x + a_2 y + a_3 z \\ \frac{dy}{dt} &= \\ \frac{dz}{dt} &= \end{aligned} \right)$$

see 3b1b "How (and why) to raise e to the power of a matrix"

+ MSE Arturo Magidin's answer to "What is importance of eigenvalues/eigenvectors"

... hint, deep connection b/t recursive formula like $F_n = F_{n-1} + F_{n-2}$ and ODE's.

Intuition: $\underbrace{F_n - F_{n-1}}_{\text{"derivative of Fibonacci"}} = F_{n-2}$

Sort of look like differential equation???

Resources: MSE "Relation b/t differential equations and sequence recursions"

↳ generating functions.

Moral: when want to understand evolution of dynamical systems (recurrence relations, ODE's, Markov Chains), strategy is to understand things/vectors/objects that transformation acts on in simplest possible way and write everything in terms of those simplest possible building blocks.

Discussion 7/26/22

Google 115A practice finals:

Practice 115A Finals

With Solutions:

https://www.math.ucla.edu/~tao/resource/general/115a.3.02f/final_solutions.pdf

https://www1.cmc.edu/pages/faculty/Bhunter/math115a/final-practice-solutions.pdf

https://jenseberhardt.com/teaching/F17-115Adata/practicefinalsolution.pdf

Without Solutions: actually some (problems 1-6) solutions, go to URL, and look around.

https://www.math.ucla.edu/~tao/resource/general/115a.3.02f/115aw.pdf

Today I try to go over all of ↑

↓

Exercise 7: Regarding complex #s $\mathbb{C}$ as a $\mathbb{R}$-VS

define $\langle z_1,z_2\rangle = \frac{1}{2}(z_1\bar{z}_2 + z_2\bar{z}_1)$.

(a): show this is inner product.

- $\boxed{\mathbb{F}\text{-valued}}$: so want to show for all $z_1,z_2\in\mathbb{C}$, $\langle z_1,z_2\rangle \in \mathbb{R}$

  well, for sure it is in $\mathbb{C}$. (b/c sum and prod of $\mathbb{C}$ #s is in $\mathbb{C}$)

  To show in $\mathbb{R}$, show $\langle z_1,z_2\rangle = \overline{\langle z_1,z_2\rangle}$ $\boxed{\text{(b/c for } z\in\mathbb{C},\ z=\bar{z} \Leftrightarrow z\in\mathbb{R})}$.

  well,
  $$\overline{\frac{1}{2}(z_1\bar{z}_2 + z_2\bar{z}_1)} = \frac{1}{2}(\overline{z_1\bar{z}_2} + \overline{z_2\bar{z}_1})$$
  $$= \frac{1}{2}(\bar{z}_1 z_2 + \bar{z}_2 z_1).$$
  $$= \langle z_1,z_2\rangle$$

  (using complex conjugation commutes w/ addition, mult).

- $\boxed{\text{Linearity}}$:
  - • Additivity: $\langle z_1+z_2, w\rangle := \frac{1}{2}\left((z_1+z_2)\bar{w} + w\overline{(z_1+z_2)}\right)$

    $= \dots = \langle z_1,w\rangle + \langle z_2,w\rangle$.
  - • scalar multiplication:

    $\langle cz, w\rangle := \frac{1}{2}\left((cz)\bar{w} + w\overline{(cz)}\right)$. where $c\in\mathbb{R}$!

    $(z,w\in\mathbb{C})$.

    $= \dots = c\langle z,w\rangle$.

- $\boxed{\text{Conjugate:}}$ $\langle z_1,z_2\rangle \overset{\text{easy to check by hand}}{=} \langle z_2,z_1\rangle = \overline{\langle z_2,z_1\rangle}$ both $\mathbb{R}$-valued.

- $\boxed{\text{Zero condition:}}$ if $z\neq 0$, then $\langle z,z\rangle$ is positive $\mathbb{R}$ #.

  $$\langle z,z\rangle = \frac{1}{2}(z\bar{z} + \bar{z}z) = \frac{1}{2}(|z|^2 + |z|^2) = |z|^2 > 0.$$

Tip, when writing proof to check many conditions, be very clear and format/layout your answer well (bullet points, boxes, indentation, etc).

(b) If $z = a+bi$, $w = c+di$ WTS

$$\langle z, w \rangle = ac + bd.$$

Pf: just do the symbol pushing.

Note, this is exactly dot product on $\mathbb{R}^2$. (identifying $\overset{\in \mathbb{C}}{(a+bi)} \leftrightarrow (a,b) \in \mathbb{R}^2$, $(c+di) \leftrightarrow (c,d) \in \mathbb{R}^2$)

So, if you know that dot product is inner product, you could do (a) by doing (b) first, and then say "dot product on $\mathbb{R}^2$ is inner product).

(c) let $\overset{a \in \mathbb{C}}{M_a} : \mathbb{C} \to \mathbb{C}$ be defined by $M_a(z) := az$. (Check: M_a linear).

WTS $\langle M_a(z_1), M_a(z_2) \rangle = a\bar{a} \langle z_1, z_2 \rangle$.

$$\overset{\cdot\cdot}{\|}$$

commutativity of $\mathbb{C}$-mult.

$$\frac{1}{2}\left((az_1)(\overline{az_2}) + (\overline{az_1})(az_2)\right) = \frac{1}{2}\left(a\bar{a} z_1 \bar{z}_2 + \bar{a} a \bar{z}_1 z_2\right)$$

$$= \frac{1}{2} a\bar{a} (z_1\bar{z}_2 + \bar{z}_1 z_2) =: a\bar{a} \langle z_1, z_2 \rangle$$

(d) $\boxed{\text{WTS}}$ M_a is $\overset{\text{same}}{\text{iso}}\underset{\text{distance}}{\text{metry}} \Leftrightarrow |a| = 1$.

that is to say $\underset{\uparrow}{|M_a(z)|} = \underset{\uparrow}{|z|}$ $\forall z \in \mathbb{C}$.

defined in terms of $\langle \cdot, \cdot \rangle$

i.e. $\langle M_a(z), M_a(z) \rangle = \langle z, z \rangle$.

Pf: ($\Rightarrow$): assume isometry. Then, from part (C),

$$\underbrace{a\bar{a}}_{|a|^2} \langle z, z \rangle = \langle z, z \rangle.$$

If $z \neq 0$, we know $\langle z, z \rangle > 0$ so can divide:

$$\underbrace{a\bar{a}}_{|a|^2} = 1 \Rightarrow |a| = 1 \quad (\text{b/c } |a| > 0).$$

two solutions to $x^2 = 1$

$(\Leftarrow)$: $|a|=1$, so $a\bar{a}=|a|^2=1$, so indeed

$$\langle M_a(z), M_a(z)\rangle \underset{\overset{\uparrow}{(c)}}{=} a\bar{a}\langle z, z\rangle = \langle z, z\rangle.$$ as desired.

(e) if $T:\mathbb{C}\to\mathbb{C}$ defined by $T(z):=\bar{z}$, then T is isometry.

WTS $\langle Tz, Tz\rangle = \langle z, z\rangle$

but $\langle Tz, Tz\rangle := \langle \bar{z}, \bar{z}\rangle := \frac{1}{2}(\bar{z}\bar{\bar{z}} + \bar{\bar{z}}\bar{z})$

$= \frac{1}{2}(\bar{z}z + z\bar{z}) =: \langle z, z\rangle.$

Exercise 8: U be $\overset{\text{column are ON-basis (orthonormal)}}{\boxed{\text{orthogonal matrix}}}$. WTS $\overset{\text{the following are equivalent}}{\text{TFAE}}$

① U is symmetric ($U=U^T$)

② $U^2=I$.

Example
if $T:V\to V$ is map b/t f.d VS,

TFAE:
- T inj
- T surj
- T bij
- T invertible
- $\det T \neq 0$
- T maps basis to basis

Copy paste from above!

Algebraic property of ON-matrices

$$\underset{U^T}{\begin{bmatrix} — v_1 — \\ \vdots \\ — v_n — \end{bmatrix}} \underset{U}{\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}} = \begin{pmatrix} \overset{\text{dot products}}{\overset{=\|v_1\|^2}{\langle v_1, v_1\rangle}} & \langle v_1, v_2\rangle & \\ \langle v_2, v_1\rangle & \ddots & \\ \vdots & & \ddots \end{pmatrix}$$

$\| \iff$ v_i's are ON basis

$$\rightarrow \begin{pmatrix} 1 & 0 & & \\ 0 & 1 & & 0 \\ & & 1 & \\ & 0 & & \ddots \end{pmatrix} = I$$

① ⇒ ②: if $U=U^T$, and we know $U^TU=I$, Then,

$U^2 = U\cdot U = U^TU = I$.

② ⇒ ①: if $U^2=I$, that means U is invertible, and $U^{-1}=U$.

By uniqueness of inverse:

$$U^T = U^T I = \underline{U^T(UU) = U^T U U = (U^T U) U} = I \cdot U = U.$$

sort of clever.
but **TIP**: all uniqueness proofs sort of proceed this way,
recall uniqueness of additive inverse in
like § 1.1.

OH 7/26/22

Exercise 1 of Jens Eberhardt:

$T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $T(f) := f(1) + f(-1)\,x + f(0)\,x^2$.

i.e., if $f = ax^2 + bx + c$. Then

$$T(ax^2+bx+c) := \underbrace{[a+b+c]}_{f(1)} + \underbrace{[a-b+c]}_{f(-1)}\,x + \underbrace{[c]}_{f(0)}\,x^2$$

Example:

$$T(x^2+5x+7) = 13 + 3x + 7x^2.$$

Midterm Q6 4:

$S := v_1 \cdots v_3$ is LD, $\vec{0} \notin S$. Does there $\exists$ $a_1, a_2 \in F$ s.t.

$a_1 v_1 + a_2 v_2 = v_3$. **NO**

well, LD just says $\exists$ $a_1, a_2, a_3 \in F$, not **all** 0 s.t.

$$a_1 v_1 + a_2 v_2 + a_3 v_3 = 0.$$

If $a_3 \neq 0$, then $v_3 = \left(-\frac{a_1}{a_3}\right) v_1 + \left(-\frac{a_2}{a_3}\right) v_2$, then yes

b/c division!

If $a_3 = 0$ then we can not do $\uparrow$.

↳ v_1 and v_2 are scalar multiples of each other.

Narrow the search.

Now to produce counter example: consider $\mathbb{R}^2$

$v_1 = (1,0)$ $v_2 = (2,0)$

$v_3 = (0,1)$.

Another perspective:

$\underbrace{a_1 v_1 + a_2 v_2}_{\text{span}\{v_1, v_2\}} = \underbrace{v_3}_{\text{span}\{v_3\}}$.

So, question becomes: can $v_1 v_2 v_3$ be LD but yet have $\text{span}\{v_3\} \not\subseteq \text{span}\{v_1, v_2\}$.

LD $\Rightarrow$ $\text{span}\{v_1, v_2, v_3\}$ has dimension ≤ 2.

But if $v_3 \notin \text{span}\{v_1, v_2\}$, then

$\dim \text{span}\{v_1 .. v_3\} = 1 + \dim \text{span}\{v_1, v_2\}$.

$\Rightarrow$ $\dim \text{span}\{v_1, v_2\} \leq 1$.

Midterm Q6 Prob 9 : $\overset{?}{\exists}$ VS V, $T: V \to V$ surj but T not injective.

Fact : if V is f.d., then recall ____

So if $\exists$ counterexample, must be ∞-dim

Counter Example : the VS of all polynomials $\overset{\text{over } \mathbb{R}}{}$. $\mathbb{R}[x]$ is $\mathbb{R}$-VS.

Then derivative map $\frac{d}{dx} : \mathbb{R}[x] \to \mathbb{R}[x]$ is surjective!

B/c for any f in codomain $\mathbb{R}[x]$, take formal antiderivative to get $F \in \mathbb{R}[x]$ s.t. $\frac{d}{dx} F = f$.

But not injective b/c all $c \in \mathbb{R} \xmapsto{\frac{d}{dx}} 0$.

§6.2 Prob 13

Suppose already proven $S \subseteq (S^{\perp})^{\perp}$. WTS $\text{span}\, S \subseteq (S^{\perp})^{\perp}$.

This comes from fact that $\forall$ set $M \subseteq V$, $M^{\perp}$ is subspace of V.

"easily seen" according to pg 347.

Tip : If subspace contains subset, then subspace contains $\underbrace{\text{subspace generated by that subset}}_{\text{exactly span.}}$

Midterm Q2: V VS, W subspace $\subseteq V$

1. WTS $x+W$ is subspace $\Leftrightarrow$ $x \in W$.

($\Rightarrow$): $x+W$ subspace $\Rightarrow$ $0 \in x+W$. By def, $\exists\, w \in W$ st. $0 = x+w$.

$\Rightarrow$ $w = -x$ $\Rightarrow$ $-x \in W \Rightarrow$ $x \in W$.

$\uparrow$ mult by -1.

($\Leftarrow$): Assuming $x \in W$
Claim $x+W = W$, hence $x+W$ is subspace.

To show this, ($\supseteq$) and ($\subseteq$).

($\subseteq$): arbitrary element of $x+W$ looks like $x+w$ for $w \in W$.

But $x, w \in W \Rightarrow$ $x+w \in W$, hence all elements of $x+W$ are $\subseteq W$.

($\supseteq$): Suppose arbitrary $w \in W$. Then $x \in W \Rightarrow -x \in W \Rightarrow w - x \in W$.

Then

$$w = \underbrace{x + \underbrace{(w-x)}_{\in W}}_{\in x+W}$$

, so indeed all $w \in x+W$.

Midterm Stats:

Mean: 53.63 (/61).

median: 55

Max: 61

min: 28

stdev: 7.12

Midterm Q3 from scratch:

(a) WTS β_a is basis.

Spanning. let $v \in V$ be arbitrary. Then b/c $v_1 \cdots v_n$ is basis, $\exists\, a_1 \cdots a_n \in F$ st

$\boxed{v = \sum a_i v_i.}$ Then, $\boxed{v = \sum \left(\frac{a_i}{a}\right) \cdot (a v_i)}$ hence, (arbitrary) v is linear combo of β_a vectors.

LI: suppose arbitrary linear combo of β_a vectors $a_1(av_1) + \cdots + a_n(av_n) = 0$.

$$= (a_1 a) v_1 + \cdots + (a_n a) v_n = 0$$

B/c $v_1 \cdots v_n$ is LI, all $a_i a = 0 \Rightarrow$ all $a_i = 0$. ✓

(b) WTS β_{+v_1} is basis

Spanning: let $v \in V$ be arbitrary. Then b/c $v_1 \cdots v_n$ is basis, $\exists\, a_1 \cdots a_n \in F$ st

$\boxed{v = \sum a_i v_i.}$ Then want

$$v = b_1(v_1+v_1) + b_2(v_2+v_1) + \cdots + b_n(v_n+v_1).$$

$$= (2b_1 + b_2 + \cdots + b_n)v_1 + b_2v_2 + \cdots + b_nv_n.$$

i.e. would have to have
$$a_1 = 2b_1 + b_2 + \cdots + b_n$$
$$b_2 = a_n$$
$$\vdots$$
$$b_n = a_n$$

That is to say defining $\boxed{\begin{matrix} b_2 := a_2 \\ \vdots \\ b_n := a_n \end{matrix} \quad \text{and} \quad b_1 := \frac{a_1 - a_2 - \cdots - a_n}{2}}$

we indeed have $v = \sum a_i v_i \Rightarrow \boxed{v = \sum b_i(v_i + v_1).}$

So arbitrary $v \in V$ is indeed linear combo. of β_{+v_1} vectors.

LI: suppose $\overset{\text{arbitrary linear combo of } \beta_{+v_1} \text{ vectors}}{\vee}$
$$a_1(\underline{v_1+v_1}) + \cdots + a_n(\underline{v_n+v_1}) = 0$$
$$\shortparallel$$
$$(2a_1 + a_2 + \cdots a_n)v_1 + a_2v_2 + \cdots + a_nv_n = 0.$$

B/c $v_1 \cdots v_n$ is LI, we must have
$$2a_1 + a_2 + \cdots + a_n = 0$$
$$a_2 = 0$$
$$\vdots$$
$$a_n = 0$$

But $\underbrace{a_2 = \cdots = a_n = 0, \text{ plus } 2a_1 + a_2 + \cdots + a_n = 0}$
$$\Rightarrow 2a_1 = 0 \Rightarrow a_1 = 0.$$

Hence, arbitrary linear combo is trivial.

Even shorter: (using cardinality arguments!)

Midterm Q3 from scratch:

(a) WTS β_a is basis.

Spanning. Let $v \in V$ be arbitrary. Then b/c $v_1 \cdots v_n$ is basis, $\exists a_1 \cdots a_n \in F$ s.t. $v = \sum a_i v_i$. Then, $v = \sum \left(\frac{a_i}{a}\right)\cdot(av_i)$ hence $\overset{\wedge}{}$ v is linear combo. of β_a vectors.
(arbitrary)

LI: b/c β_a is spanning, has n vectors, where $n = \dim V$, it must also be LI (b/c if LD, could delete/remove until get smaller $\underset{\text{basis}}{\underline{\text{LI spanning set}}}$, but that's impossible since all bases have same size/cardinality).

(b) WTS β_{+v_1} is basis

LI: suppose $\overset{\text{arbitrary linear combo of } \beta_{+v_1} \text{ vectors}}{\vee}$
$$a_1(v_1+v_1) + \cdots + a_n(v_n+v_1) = 0$$
$$\shortparallel$$
$$(2a_1 + a_2 + \cdots a_n)v_1 + a_2v_2 + \cdots + a_nv_n = 0.$$

B/c $v_1 \cdots v_n$ is LI, we must have
$$2a_1 + a_2 + \cdots + a_n = 0$$
$$a_2 = 0$$

$$\vdots$$

$$a_n = 0$$

But $\underbrace{a_2 = \cdots = a_n = 0, \quad \text{plus} \quad 2a_1 + a_2 + \cdots + a_n = 0}$

$\Rightarrow 2a_1 = 0 \Rightarrow a_1 = 0.$

Hence, arbitrary linear combo is trivial.

Spanning: $\underbrace{|\beta + v_1| = |\beta|, \quad \beta + v_1 \text{ is LI}}$

$\Rightarrow$ similar to above argument, basis.