246A Homework 8

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Exercise 40 (Notes 4)

(Inverse function theorem for holomorphic functions) Before we begin this exercise (which is to prove the theorem stated in the first *Theorem* box below), let us review some relevant results from introductory real analysis (the impatient reader may skip to the problem below starting at the subsection titled **The Actual Exercise**).

Theorem: Inverse function theorem for holomorphic functions

Let $U \subseteq \mathbb{C}$ be open, $z_0 \in U$, and let $f : U \to \mathbb{C}$ be holomorphic satisfying $f'(z_0) \neq 0$. Then there exists a neighborhood V of z_0 in U s.t. the map $f : V \to f(V)$ is a complex diffeomorphism (also called biholomorphic), i.e. f is holomorphic, and has an inverse f^{-1} that is also holomorphic (which additionally must satisfy $(f^{-1})'(w) = \frac{1}{f'(f^{-1}(w))}$ by the chain rule).

The relevant results from introductory real analysis are of course the implicit and inverse function theorems, which I shall state now:

Lemma: Implicit function theorem (single equation)

The simplest case is if we have a $\mathcal{C}^1$ function $F(\mathbf{x}, y) : \Omega \to \mathbb{R}$ for some open neighborhood $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}$ containing $(\mathbf{a}, b) \in \mathbb{R}^n \times \mathbb{R}$ where $[\partial_y F](\mathbf{a}, b) \neq 0$; then there exist open neighborhoods $U \subseteq \mathbb{R}^n$ of $\mathbf{a}$ and $V \subseteq \mathbb{R}$ of b (could take U, V to be balls centered at $\mathbf{a}$ and b resp.) s.t. for every $\mathbf{x} \in U$, there is a unique $y \in V$ (denote this $f(\mathbf{x})$) s.t. $F(\mathbf{x}, f(\mathbf{x})) = 0$, where furthermore f is $\mathcal{C}^1$, meaning we can apply the chain rule which gives us $[\partial_{x_j} f](\mathbf{x}) = -\frac{[\partial_{x_j} F](\mathbf{x}, f(\mathbf{x}))}{[\partial_y F](\mathbf{x}, f(\mathbf{x}))}$.

From the formulas for the first order partial derivatives, it is clear that if F was originally $\mathcal{C}^k$ ($k \in \{1, 2, \ldots, \infty\}$), then the resulting f is too (the same is true for the below generalization). The following generalization follows from a somewhat involved induction on m, using the induction hypothesis for $m - 1$ and reducing down to something in the jurisdiction of the simplest case (see the paragraph after the below *Theorem* box for some references):

Theorem: Implicit function theorem (system of equations)

More generally, we have that for a $\mathcal{C}^k$ function $\mathbf{F}(\mathbf{x}, \mathbf{y}) : \Omega \to \mathbb{R}^m$ for some open neighborhood $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}^m$ containing $(\mathbf{a}, \mathbf{b}) \in \mathbb{R}^n \times \mathbb{R}^m$ where $\det([\partial_{\mathbf{y}} \mathbf{F}](\mathbf{a}, \mathbf{b})) \neq 0$ (where $\partial_{\mathbf{y}} \mathbf{F}$ denotes the $m \times m$ matrix with (i, j)th element — ith row, jth column — equal to $\partial_{y_j} F_i$), there exist open neighborhoods $U \subseteq \mathbb{R}^n$ of $\mathbf{a}$ and $V \subseteq \mathbb{R}^m$ of $\mathbf{b}$ s.t. for every $\mathbf{x} \in U$, there is a unique $\mathbf{y} \in V$ (denote this $\mathbf{f}(\mathbf{x})$) s.t. $F(\mathbf{x}, \mathbf{f}(\mathbf{x})) = 0$, where furthermore $\mathbf{f}$ is $\mathcal{C}^k$ and has Jacobian ($\in \mathbb{R}^{m \times n}$

— m rows, n columns) $[J\mathbf{f}](\mathbf{x}) := [\partial_{\mathbf{x}}\mathbf{f}](\mathbf{x}) = -\big[[\partial_{\mathbf{y}}\mathbf{F}](\mathbf{x}, \mathbf{f}(\mathbf{x}))\big]^{-1}\big[[\partial_{\mathbf{x}}\mathbf{F}](\mathbf{x}, \mathbf{f}(\mathbf{x}))\big]$ (as can be found via the chain rule).

There's a nice YouTube video about the simplest $F(x, y) : \mathbb{R}^2 \to \mathbb{R}$ case (using the "I-diagram" and IVT for existence), and Folland (Thm. 3.1, Thm. 3.9, Thm. 3.18) has a decently friendly exposition (mentioning that the intuition behind the implicit function theorem's non-zero derivative or invertible Jacobian matrix condition is that that's precisely the condition needed in the case that F is some linear/affine function $L(\mathbf{x}, \mathbf{y}) = A\mathbf{x} + B\mathbf{y} + \mathbf{c}$, and of course differentiable functions should carry the intuition of being "locally linear"). This Arxiv paper presents a pretty thorough treatment of both theorems (starting from implicit and using that to prove inverse, exactly like Folland except in a more condensed paper format).

Theorem: Inverse function theorem

If $\mathbf{f} : \Omega \to \mathbb{R}^n$ is $\mathcal{C}^1$ for some open neighborhood $\Omega \subseteq \mathbb{R}^n$ containing $\mathbf{a} \in \mathbb{R}^n$ where $\det([J\mathbf{f}](\mathbf{a})) \neq 0$ (where the Jacobian matrix $J\mathbf{f} := \partial_{\mathbf{x}}\mathbf{f}$ denotes the $n \times n$ matrix with (i, j)th element — ith row, jth column — equal to $\partial_{x_j} f_i$), then there exist open neighborhoods $U \subseteq \mathbb{R}^n$ of $\mathbf{a}$ and $V \subseteq \mathbb{R}^n$ of $f(\mathbf{a})$ s.t. $\mathbf{f}$ is a bijective map $U \hookrightarrow\!\!\!\!\rightarrow V$, where furthermore the inverse map $\mathbf{f}^{-1}$ is $\mathcal{C}^1$ and has Jacobian ($\in \mathbb{R}^{m \times n}$ — m rows, n columns) $[J(\mathbf{f}^{-1})](\mathbf{f}(\mathbf{x})) := \big[[J\mathbf{f}](\mathbf{x})\big]^{-1}$ for every $\mathbf{x} \in U$ (as can be found via the chain rule). From the formulas for the Jacobian matrix, it is clear that if $\mathbf{f}$ was originally $\mathcal{C}^k$ ($k \in \{1, 2, \ldots, \infty\}$), then the resulting $\mathbf{f}^{-1} : V \hookrightarrow\!\!\!\!\rightarrow U$ is too.

The inverse function theorem follows quite straightforwardly from the implicit function theorem, namely for $f : \mathbb{R}^n \to \mathbb{R}^n$, we can define $\mathbf{F} : \mathbb{R}^{n+n} \to \mathbb{R}^n$ by $\mathbf{F}(\mathbf{x}, \mathbf{y}) = f(\mathbf{y}) - \mathbf{x}$ (see YouTube video or Thm. 3.18 in Folland for more details); and conversely, the implicit function theorem follows in decently short order from the inverse function theorem (given $\mathbf{F} : \mathbb{R}^{n+m} \to \mathbb{R}^m$, we want $\mathbf{f} : \mathbb{R}^m \to \mathbb{R}^m$ s.t. $\mathbf{F}(\mathbf{x}, \mathbf{f}(\mathbf{x})) = 0$, and indeed we can construct such an $\mathbf{f}$ by first defining $\mathbf{H}(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, \mathbf{F}(\mathbf{x}, \mathbf{y}))$ mapping $\mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$, and cooking up its local inverse $\mathbf{G}(\mathbf{x}, \mathbf{y}) = (\mathbf{G}_n(\mathbf{x}, \mathbf{y}), \mathbf{G}_m(\mathbf{x}, \mathbf{y}))$ — $\mathbf{G}_n, \mathbf{G}_m$ respectively mapping $\mathbb{R}^{n+m} \to \mathbb{R}^n$ and $\mathbb{R}^m$ — so that $(\mathbf{x}, \mathbf{y}) = \mathrm{id}_{\mathbb{R}^{n+m}} = \mathbf{H}(\mathbf{G}(\mathbf{x}, \mathbf{y})) = \mathbf{H}(\mathbf{G}_n(\mathbf{x}, \mathbf{y}), \mathbf{G}_m(\mathbf{x}, \mathbf{y})) = \big(\mathbf{G}_n(\mathbf{x}, \mathbf{y}), \mathbf{F}\big(\mathbf{G}_n(\mathbf{x}, \mathbf{y}), \mathbf{G}_m(\mathbf{x}, \mathbf{y})\big)\big) \implies \mathbf{x} = \mathbf{G}_n(\mathbf{x}, \mathbf{y}) \implies \mathbf{y} = \mathbf{F}(\mathbf{x}, \mathbf{G}_m(\mathbf{x}, \mathbf{y})) \implies \mathbf{0} = \mathbf{F}(\mathbf{x}, \mathbf{G}_m(\mathbf{x}, \mathbf{0}))$, i.e. $\mathbf{f}(\mathbf{x}) = \mathbf{G}_m(\mathbf{x}, \mathbf{0})$ suffices).

Interestingly, the "standard proof" (Wiki link) of the inverse function theorem looks nothing like the "standard proof" of the implicit function theorem. The gist is to use the Banach contraction mapping principle to construct the inverse, and then do a bit (like actually just a bit, as seen in the above Wiki link) of work to show that the inverse is $\mathcal{C}^1$ (this proof has the added benefit of working in any Banach space) — though to extend to C^k there seems to be some induction relying on the map $[A \mapsto A^{-1}]$ is C^∞ (which according to that Wiki link, *"is an elementary fact because the inverse of a matrix is given as the adjugate matrix divided by its determinant"*).

For the even more general versions of these theorems for Banach spaces, see these lecture notes of

Bruce K. Driver; Thm. 22.26 and Thm. 22.27 are respectively the inverse and implicit function theorems (further noteworthy things include the "converse chain rule" in Thm. 22.6 used to upgrade continuous f to differentiable f assuming that g and $g \circ f$ are differentiable, and Example 22.18 or MSE showcasing the quite messy computations involved in proving that $[A \mapsto A^{-1}]$ is infinitely Frechét differentiable, in the sense defined for Banach spaces).

The Actual Exercise

One can use the above real-variable inverse function theorem to give a short proof: if $f : U \to \mathbb{C}$ is holomorphic, it can be thought of as an infinitely differentiable (in particular $\mathcal{C}^1$) function $\mathbb{R}^2 \to \mathbb{R}^2$ (satisfying the Cauchy-Riemann equations of course), and so the real-variable inverse function theorem tells us that indeed there is some neighborhood V of z_0 such that the restriction $f : V \hookrightarrow\!\!\!\!\rightarrow f(V)$ has an inverse $f^{-1} : f(V) \hookrightarrow\!\!\!\!\rightarrow V$ that is $\mathcal{C}^1$ (Frechét differentiable). And of course for Frechét differentiable functions, complex differentiability simply comes down to satisfying the Cauchy-Riemann equations, and those can be checked quite easily (a less computational — but less rigorous — way of verification is that Cauchy-Riemann $\iff$ Jf is a rotation/dilation matrix, and of course Jf being a rotation/dilation matrix implies that $J(f^{-1}) = \left[Jf\right]^{-1}$ is also a rotation/dilation matrix).

We also provide an approach more centered on complex-analytic methods. The construction of the set inverse is based on Rouche's theorem (basically a more quantified version of our proof of the open mapping theorem, using the additional information that $f'(z_0) \neq 0$). First let us recall the statement of Rouche's theorem: if $f, g : U \to \mathbb{C}$ are holomorphic and U contains the image and interior of a simple closed curve γ where $|g(z)| < |f(z)|$ for all $z \in \mathrm{im}(\gamma)$, then f and $f + g$ have the same number of zeroes (counting multiplicity) in $\mathrm{int}(\gamma)$.

Because the zeroes of an analytic function are isolated, there is some disk $D(z_0, r_+)$ s.t. z_0 is the only root of $f(z) - w_0$ (where $w_0 := f(z_0)$) in $D(z_0, r_+)$. And since $f'(z_0) \neq 0$, the "a_1 coefficient" of the Taylor expansion of $f(z) - w_0$ at z_0 is non-zero, meaning the order of the zero is 1. By Rouche's theorem applied to $\gamma_{z_0,r,\circlearrowleft}$, for all w_1 close enough to w_0 (more explicitly, $|w_1 - w_0| < c$ where $c := \min_{z \in \partial D(z_0,r)} |f(z) - w_0|$, which is positive because continuous functions on a compact sets attain their minimums, and $|f(z) - w_0|$ is positive on $\partial D(z_0, r) \subseteq D(z_0, r_+) \setminus \{z_0\}$), we have that $f(z) - w_1$ has exactly one zero in $D(z_0, r)$, which we denote by $h(w_1)$. Observe that $h(w_1)$ being a zero of $f(z) - w_1 \implies f(h(w_1)) = w_1$.

Defining V to be the inverse image $V := f^{-1}(D(w_0, c)) \subseteq D(z_0, r)$ (which is open by the continuity of f), we have that for any $z \in V$, $h(f(z)) = z$, because by definition of V, z is an element of $D(z_0, r)$ s.t. $f(z) =: w \in D(w_0, c)$, and we know from the previous paragraph that such an element is unique and denoted by $h(w)$. Combining this with the above fact that $f(h(w_1)) = w_1$ for all $w_1 \in D(w_0, c)$, we have shown that $f : V \to D(w_0, c)$ has inverse $h : D(w_0, c) \to V$, and so $f(V) = D(w_0, c)$ and we have found our locally invertible function $f : V \to f(V)$.

Now to prove the holomorphicity of $f^{-1} := h : D(w_0, c) \to V$. Note that the open mapping theorem already tells us that h is continuous; we will use this to prove that h is holomorphic. Fix $w_1 \in D(w_0, c)$ and $z_1 := h(w_1)$ (for of course the original z_0 and $w_0 := f(z_0)$). Taylor expansion of f at z_1 yields $f(z) - f(z_1) = \sum_{n=1}^{\infty} a_n (z - z_1)^n$, where $a_1 = f'(z_1)$ is non-zero because we know from the above application of Rouche's theorem that $f(z) - f(z_1)$ has exactly one zero in $D(z_0, r)$ counting multiplicity. So $f(z) - f(z_1) = (z - z_1) g(z)$ for some g holomorphic locally at z_1 (say within ϵ of z_1) and equals $a_1 \neq 0$ at z_1.

Then (using continuity of h): $|w - w_1| < \delta \implies |h(w) - h(w_1)| < \epsilon \implies f(h(w)) - f(h(w_1)) = \big(h(w) - h(w_1)\big)g(h(w)) \implies w - w_1 = \big(h(w) - h(w_1)\big)g(h(w)) \implies \frac{h(w)-h(w_1)}{w-w_1} = \frac{1}{g(h(w))}$ (for $w \neq w_0$), and as g is a non-zero holomorphic function locally at $z_1 = h(w_1)$, the quotient $\frac{1}{g(h(w))}$ is a holomorphic function locally at z_1, and so the limit $\lim_{w \to w_1} \frac{h(w)-h(w_1)}{w-w_1}$ exists and equals $\frac{1}{g(z_1)}$. Finally, recall from our definition of g that $g(z_1) = a_1 = f'(z_1)$, so indeed $h'(w_1) = \frac{1}{f'(h(w_1))}$, and we are done.

Bonus Content

One final method of proving this is to use Rouche's theorem (as above) to prove the existence of the inverse, and to use the argument principle to construct an integral formula for the inverse (thereby proving that the inverse must be holomorphic by say Ex. 36(OLD#)/38(NEW#) "integrals of holomorphic functions are holomorphic" of Notes 3, or the geometric series integral-sum interchange trick used in Cor. 18(OLD#)/20(NEW#) of Notes 3 to prove from the Cauchy integral formula that holomorphic functions are analytic).

Here are the details: Rouche's (as above) gives us that $f^{-1}(w_1) \in D(z_0, r)$ is the unique zero of $f(z) - w_0$ for $z \in D(z_0, r_+)$, and the argument principle tells us that $\frac{1}{2\pi i} \oint_{\gamma_{z_0,r,\circlearrowleft}} \frac{f'(z)}{f(z)-w_1}\, dz$ counts the # of zeroes −# of poles (counting multiplicity) for $f(z) - w_1$ in $D(z_0, r)$. We know that $f(z) - w_1$ has a unique zero in $D(z_0, r)$ (and is analytic on U so it has no poles), but what is the order of that one zero? Well we have the Taylor expansion around $f^{-1}(w_1)$ is $0 + f'(f^{-1}(w_1)) \cdot (z - f^{-1}(w_1)) + \dots$ and because $f' \neq 0$ on $D(z_0, r)$ (by assumption $f'(z_0) \neq 0$ and so f' is non-zero on a neighborhood of z_0 by continuity) we do indeed have that the "a_1 coefficient" of the Taylor series is non-zero, i.e. the zero is of order 1. So $\frac{1}{2\pi i} \oint_{\gamma_{z_0,r,\circlearrowleft}} \frac{f'(z)}{f(z)-w_1}\, dz = \operatorname{Res}\Big[\frac{f'(z)}{f(z)-w_1}, f^{-1}(w_1)\Big] = 1$.

We know from the proof of the argument principle (in particular the part $F(z) = (z - z_0)^k G(z) \implies \frac{F'(z)}{F(z)} = \frac{k}{z-z_0} + \frac{G'(z)}{G(z)}$) that for holomorphic F on U, $\frac{F'(z)}{F(z)}$ always has order $\geqslant -1$, so the above integral/residue equalling 1 tells us that the Laurent expansion of $\frac{f'(z)}{f(z)-w_1}$ around $f^{-1}(w_1)$ is $\frac{1}{z-f^{-1}(w_1)} + g(z)$ (where g is some standard power series centered at $f^{-1}(w_1)$).

Then, $\frac{f'(z)}{f(z)-w_1} \cdot (z - f^{-1}(w_1))$ is holomorphic locally at $f^{-1}(w_1)$ (locally at $f^{-1}(w_1$ it equals $1 + (z - f^{-1}(w_1))g(z)$), hence holomorphic on all of $D(z_0, r_+)$ (outside of $f^{-1}(w_1)$, it is just a product/quotient of holomorphic functions). Cauchy's theorem yields $\oint_{\gamma_{z_0,r,\circlearrowleft}} \frac{f'(z)(z-f^{-1}(w))}{f(z)-w_1}\, dz = 0$, implying that

$$\oint_{\gamma_{z_0,r,\circlearrowleft}} \frac{zf'(z)}{f(z)-w_1}\, dz = f^{-1}(w_1) \cdot \oint_{\gamma_{z_0,r,\circlearrowleft}} \frac{f'(z)}{f(z)-w_1}\, dz = f^{-1}(w_1) \cdot 1.$$

Of course w_1 is arbitrary in $D(w_0, c)$ so indeed we have found an integral formula for the (local) inverse $f^{-1} : D(w_0, c) \to V$. It is sort-of surprising that we have such a clean formula for the inverse (I'm not aware of any such real-analytic integral for the inverse of a function), but even with just the knowledge that f^{-1} was holomorphic and the formula $h'(w_1) = \frac{1}{f'(h(w_1))}$, we can construct a Taylor series so we technically already had a (series) formula for the inverse (admittedly not as clean and harder to work with than the integral).

Exercise 42

Theorem: Non-vanishing and injectivity preserved by locally uniform convergence (Hurwitz)

Let $U \subseteq \mathbb{C}$ be non-empty, open, and connected, and let $f_n : U \to \mathbb{C}$ be a sequence of holomorphic functions that converge uniformly on compact sets to a limit $f : U \to \mathbb{C}$ (which must be holomorphic by Thm. 36(NEW#) of Notes 3). We have the following two claims:

(i) if none of the f_n have zeroes in U, then f either has no zeroes in U or is identically 0.

(ii) if all the f_n are univalent (i.e. injective), then either f is univalent or is constant.

Proof: note first that (i) $\implies$ (ii) by considering (for arbitrary $z_0 \in U$) the sequence $g_n(z) := f_n(z) - f_n(z_0)$ on $U \setminus \{z_0\}$, which converges locally uniformly to $f(z) - f(z_0)$. By injectivity of the f_n the functions g_n are never 0 on U, and so by (i) either $f(z) - f(z_0)$ is never 0 on U, or identically 0 (in which case f is obviously constant). Observe then that $f(z) - f(z_0)$ having no zeroes in the non-empty connected open set $U \setminus \{z_0\}$ (connected because for open sets, connected-ness and path-connectedness are equivalent, and if a path in U hits z_0 we can make a detour going around still lying in U since U open implies $D(z_0, \epsilon) \subseteq U$) means that there is no $z \neq z_0$ in U s.t. $f(z) = f(z_0)$; and as $z_0 \in U$ was taken to be arbitrary, this proves that $z \neq w \implies f(z) \neq f(w)$, and so f must indeed be injective.

Now for (i): again from Thm. 36(NEW#) of Notes 3 we know that the derivatives f_n' converge uniformly on compact sets to f', and as uniform convergence is preserved through continuous functions (such as the function $[(x_1, x_2) \mapsto \frac{x_1}{x_2}] : X \to Y$), we have that $\frac{f_n'}{f_n}$ converges uniformly to $\frac{f'}{f}$ on compact subsets of U. Then using the argument principle (and the fact that uniform convergence allows limits to be interchanged with integrals), we have that the number of zeroes of f in $D(z_0, r) \subseteq \overline{D(z_0, r)} \subseteq U$ (assuming f is not identically equal to 0 on $D(z_0, r)$, in which case by the identity theorem f would be identically 0 on all of U) equals $\oint_{\gamma_{z_0,r,\circlearrowleft}} \frac{f'(z)}{f(z)}\, dz = \lim_{n\to\infty} \oint_{\gamma_{z_0,r,\circlearrowleft}} \frac{f_n'(z)}{f_n(z)}\, dz = 0$, and so indeed f is never 0 in U.

Exercise 43

Recall that the open mapping theorem basically says that if we have non-constant holomorphic function $f : U \to \mathbb{C}$, then for every $D(z_0, r) \subseteq U$, $f(D(z_0, r))$ contains some open disk $D(f(z_0), \epsilon)$. Bloch's theorem quantifies that as follows:

Theorem: Quantified version of open mapping theorem (Bloch)

Let $f : D(z_0, R) \to \mathbb{C}$ be holomorphic and suppose $f'(z_0) \neq 0$. Then there is an absolute constant $c > 0$ ("Bloch's constant") s.t. $f(D(z_0, R))$ contains a disk of radius $c|f'(z_0)|R$.

Intuitively, if we think of $f'(z_0) \neq 0$ that tells us that locally at z_0, f is the linear (rotation/dilation) transformation $L(z) = f(z_0) + f'(z_0) \cdot (z - z_0)$, so f dilates the disk $D(z_0, \epsilon)$ approximately into $D(f(z_0), |f'(z_0)|\epsilon)$. Bloch tells us this is the case, *up to* an absolute constant $c > 0$ and a bit of perturbations off-center. We split this proof into two steps.

Step 1: if $|f'(z)| \leqslant 2|f'(z_0)|$ for all $z \in D(z_0, R)$, we show that there is an absolute constant $c > 0$ s.t. $f(D(z_0, R))$ contains the disk $D(f(z_0), c|f'(z_0)|R)$ (i.e. we still have to "pay the cost" of the constant $c > 0$, but this derivative bound condition saves us the need to shift off-center). Terry gives the following hints: we can normalize/simplify to the case $z_0 = 0, R = 1, f'(z_0) = 1$, then use the higher order Cauchy integral formula to get a bound on $f''(z)$ near the origin, and then use this to approximate f near the origin, and then finally apply Rouche's theorem, which tells us in particular that if $c < |f(z)|$ for all $z \in \partial D(0, \delta)$, then $f(z)$ and $f(z) - w_1$ (for $|w_1| < c \iff w_1 \in D(0, c)$) have the same number of zeroes (counting multiplicity) in $D(0, \delta)$, and since $f(0) = 0$, we have indeed $D(0, c) \subseteq \operatorname{im}(f) = f(D(0, 1))$.

We flesh this outline out now. For the normalization step, assuming we have $\tilde{f}$ on $D(0, 1)$ with $f(0) = 0$, $\tilde{f}'(0) = 1$ and $\tilde{f}(D(0, 1)) \supseteq D(0, c)$, we can transform to f by $f(z) = Rf'(z_0)\tilde{f}(\frac{z - z_0}{R}) + f(z_0)$ (checking that indeed the value at z_0 is $f(z_0)$ and derivative at z_0 is $Rf'(z_0)\frac{1}{R}\tilde{f}'(0) = f'(z_0)$), i.e. we can define $\tilde{f}(z) := \frac{f(Rz + z_0) - f(z_0)}{Rf'(z_0)}$, so that $f(D(z_0, R)) = Rf'(z_0)\tilde{f}(D(0, 1)) + f(z_0) \supseteq Rf'(z_0)D(0, c) + f(z_0) = D(f(z_0), R|f'(z_0)|c)$, as desired. This all means that if we can prove there is some universal constant $c > 0$ s.t. all holomorphic functions $f : D(0, 1) \to \mathbb{C}$ with $f(0) = 0$ and $f'(0) = 1$ have their image $f(D(0, 1)) \supseteq D(0, c)$, we'll be done.

Since f' is holomorphic on $D(0, 1)$, we can apply the 1st-order Cauchy integral formula on f' and the ML-estimate to get that for any (fixed) $z \in D(0, \frac{1}{2})$,

$$|f''(z)| = \left| \frac{1}{2\pi i} \int_{\gamma_{0, \frac{1}{2}, \circlearrowleft}} \frac{f'(w)}{(w - z)^2} \, dw \right| \leqslant \frac{1}{2\pi} \cdot 2\pi(\tfrac{1}{2}) \cdot \frac{2\cancel{|f'(0)|}^{1}}{(\frac{1}{2} - |z|)^2} = \frac{1}{(\frac{1}{2} - |z|)^2}.$$

Then FTC(Id) — and a very minor use of Ex. 17(iv) from Notes 2 (change of variables to Riemann integral), parameterizing the path $[0 \to z]$ by $[t \mapsto t\frac{z}{|z|}]$ for $t \in [0, |z|]$, plus Lemma 8 of Notes 2 (triangle inequality for Riemann integrals) — gives for $z \in D(0, \frac{1}{2}) \setminus \{0\}$:

$$\begin{aligned} |f'(z) - f'(0)| &= \left| \int_{[0 \to z]} f''(w) \, dw \right| = \left| \int_0^{|z|} f''(t \cdot \tfrac{z}{|z|}) \cdot \tfrac{z}{|z|} \, dt \right| \leqslant \int_0^{|z|} \left| f''(t \cdot \tfrac{z}{|z|}) \cdot \tfrac{z}{|z|} \right| dt \\ &\leqslant \int_0^{|z|} \frac{1}{(1/2 - t)^2} \, dt = \left(\tfrac{1}{2} - t\right)^{-1} \Big|_0^{|z|} = \frac{1}{1/2 - |z|} - 2. \end{aligned}$$

We can use the above bound, and the exact same techniques (FTC(Id), change of variables to Riemann integral, and triangle inequality for Riemann integrals) to bound the deviation of $f(z)$ from its local linear approximation $f'(0)\cdot(z-f(0)) = f'(0)\cdot z$ (again for $z\in D(0,\frac{1}{2})\setminus\{0\}$):

$$\begin{aligned}|f(z)-f'(0)\cdot z| &= \left|\int_{[0\to z]} f'(w)-f'(0)\,dw\right| = \left|\int_0^{|z|}\big[f'(t\cdot\tfrac{z}{|z|})-f'(0)\big]\cdot\tfrac{z}{|z|}\,dt\right| \\ &\leqslant \int_0^{|z|}\big|\big[f'(t\cdot\tfrac{z}{|z|})-f'(0)\big]\cdot\tfrac{z}{|z|}\big|\,dt \leqslant \int_0^{|z|}\frac{1}{\frac{1}{2}-|t\cdot\frac{z}{|z|}|}-2\,dt \\ &= \int_0^{|z|}\frac{1}{1/2-t}-2\,dt = \big(-\ln(\tfrac{1}{2}-t)-2t\big)\Big|_0^{|z|} = -\ln(\tfrac{1}{2}-|z|)-2|z|+\ln(\tfrac{1}{2}).\end{aligned}$$

Then, we have (by the reverse triangle inequality) that $|f(z)|\geqslant|f'(0)\cdot z|-(-\ln(\frac{1}{2}-|z|)-2|z|-\ln 2)$. For $z\in\partial D(0,\delta)\iff|z|=\delta$, this RHS is $h(\delta):=\delta+\ln(\frac{1}{2}-\delta)+2\delta+\ln 2=3\delta+\ln(1-2\delta)$. Doing some elementary calculus yields $h'(\delta)=3+\frac{-2}{1-2\delta}$ which is 0 at $3(1-2\delta)-2=0\iff\delta=\frac{1}{6}$, and $h(\frac{1}{6})=\frac{1}{2}+\ln(\frac{2}{3})\geqslant 0.0945$. In other words we have found that $|f(z)|\geqslant 0.0945$ on $\partial D(0,\frac{1}{6})$, and so looking back up at the first paragraph of **Step 1** Rouche's theorem tells us that taking $c=0.0945$ we know for certain that $D(0,c)\subseteq f(D(0,1))$.

Step 2: we now lose the additional hypothesis from **Step 1** (at the cost of a slightly worse constant $c'>0$ and maybe shift off-center), in the procedure outlined here: if $|f'(z)|\leqslant 2|f'(z_0)|$ on $D(z_0,\frac{R}{4})$, then **Step 1** applies with R replaced by $\frac{R}{4}$ (i.e. $f(D(z_0,R))\supseteq f(D(z_0,\frac{R}{4}))\supseteq D(f(z_0),c|f'(z_0)|\frac{R}{4})$, so the "slightly worse constant" is $c'=\frac{c}{4}$); if not, we can pick $z_1\in D(z_0,\frac{R}{4})$ s.t. $|f'(z_1)|>2|f'(z_0)|$, and start over with z_0 replaced by z_1 and R replaced by $\frac{R}{2}$. First I claim that this process must stop after finite time, and then I work out the new constant $c'>0$ that we get.

Ok — let us suppose f.s.o.c. that this process could continue indefinitely. Then we can find z_1 s.t. $|z_1-z_0|<\frac{R}{4}$ and $|f'(z_1)|>2|f'(z_0)|$, and then z_2 s.t. $|z_2-z_1|<\frac{R}{4^2}\implies|z_2-z_0|<R\cdot(\frac{1}{4}+\frac{1}{4^2})$ and $|f'(z_2)|>2|f'(z_1)|>2^2|f'(z_0)|$, and so on to z_n satisfying $|z_n-z_{n-1}|<\frac{R}{4^n}\implies|z_n-z_0|<R\cdot(\frac{1}{4}+\ldots+\frac{1}{4^n})<R\cdot\frac{1}{3}$. Then we have that all the $\{z_n\}_{n=0}^\infty\subseteq\overline{D(z_0,\frac{R}{3})}$, which is a compact set so by Bolzano-Weierstrass there is an accumulation point z_∞. Because f is holomorphic on $D(z_0,R)$ and we know holomorphic functions are analytic and in particular infinitely differentiable, we have that f' is continuous on U, and so the limit of $f'(z_{n_k})$ for that subsequence $z_{n_k}\to z_\infty$ exists and equals $f'(z_\infty)\in\mathbb{C}$. But this is impossible since $|f'(z_{n_k})|>2^{n_k}|f'(z_0)|\geqslant 2^k|f'(z_0)|$ goes to ∞ as $k\to\infty$, so the sequence $f'(z_{n_k})$ can not possibly converge to any complex number.

Now that we know the process must terminate at say stage N (i.e. we have $z_0,\ldots,z_N$ s.t. $|f'(z_n)|>2|f'(z_{n-1})|$ for $n\leqslant N$ and $|f(z)|\leqslant 2|f'(z_N)|$ for all $z\in D(z_N,\frac{1}{4}\cdot\frac{R}{2^N})$), we calculate the coefficient. Well since $|f(z)|\leqslant 2|f'(z_N)|$ for all $z\in D(z_N,\frac{1}{4}\cdot\frac{R}{2^N})$, **Step 1** gives that $f(D(z_N,\frac{R}{4\cdot 2^N}))\supseteq D(z_n,c|f'(z_N)|\frac{R}{4\cdot 2^N})$. But we know that $|f'(z_N)|>2|f'(z_{N-1})|>\ldots>2^N|f'(z_0)|$, so $c|f'(z_N)|\frac{R}{4\cdot 2^N}>c|f'(z_0)|\frac{R}{4}$, so even in the general stage N case the new constant c' only weakens from the **Step 1** constant c by (at most) a factor of 4! So we have found that for the general case, we may take $\frac{c}{4}=0.023625$ to be a lower bound to Bloch's (absolute) constant.

Exercise 46

We are asked to compute the integral

$$\int_0^\infty \frac{1}{\sqrt{x}(x+1)}\,dx := \lim_{\epsilon\to 0, R\to\infty} \int_\epsilon^R \frac{1}{\sqrt{x}(x+1)}\,dx$$

by applying the residue theorem to the function $\frac{b(z)}{z+1}$ where b denotes some branch of the function $[z \mapsto z^{-1/2}]$ with a branch cut *on the positive real axis*, and (small perturbations of) the (keyhole) contour $[\epsilon \to R] + \gamma_{0,R,\circlearrowleft} + [R \to \epsilon] + \gamma_{0,\epsilon,\circlearrowright}$. By exploiting the branch cut, the integrals along (perturbations of) $[\epsilon \to R]$ and $[R \to \epsilon]$ *don't* cancel each other out, allowing us to compute he desired integral.

We first describe the branch cut. Denote by $\mathrm{Log}_{\theta_0}(z) : \mathbb{C} \setminus e^{i\theta_0}\mathbb{R}_{\geqslant 0} \to \mathbb{C}$ the function $\ln|z| + i\,\mathrm{Arg}_{\theta_0}(z)$, where $\mathrm{Arg}_{\theta_0}(z) : \mathbb{C} \setminus e^{i\theta_0}\mathbb{R}_{\geqslant 0} \to \mathbb{C}$ is the unique $\theta \in [\theta_0, \theta_0 + 2\pi)$ s.t. $z = |z|e^{i\theta}$. Note that $\mathrm{Log}_{\theta_0+2\pi}(z) = \mathrm{Log}_{\theta_0} + 2\pi i$, and the "standard branch" of the logarithm Log is denoted $\mathrm{Log}_{-\pi}$ in this current notation. We then define $\mathrm{pow}_{\theta_0}(z,\alpha) : \mathbb{C} \setminus e^{i\theta_0}\mathbb{R}_{\geqslant 0} \to \mathbb{C}$ by $\mathrm{pow}_{\theta_0}(z,\alpha) := \exp(\alpha\,\mathrm{Log}_{\theta_0}(z))$. Observe that $\mathrm{pow}_{\theta_0}(z,\alpha)^{-1} = \exp(-\alpha\,\mathrm{Log}_{\theta_0}(z)) = \mathrm{pow}_{\theta_0}(z,-\alpha)$ (using $(e^z)^{-1} = e^{-z}$, a corollary of $e^{z+(-z)} = e^z e^{-z}$) and $\mathrm{pow}_{-\pi}(z,-\alpha) = \exp(\alpha \cdot -\mathrm{Log}_{-\pi}(z)) = \exp(\alpha\,\mathrm{Log}_{-\pi}(\frac{1}{z})) = \mathrm{pow}_{-\pi}(z^{-1},\alpha)$, where the middle equality is because $-\mathrm{Log}_{-\pi}(z) = -\ln|z| - i\,\mathrm{Arg}_{-\pi}(z) = \ln|\frac{1}{z}| + i\,\mathrm{Arg}_{-\pi}(\frac{1}{z})$ (we don't use this second identity, but I found it interesting that it only works in general for a specific branch). We can finally define $f_{\theta_0} : \mathbb{C} \setminus e^{i\theta_0}\mathbb{R}_{\geqslant 0}$ by $f_{\theta_0}(z) := \frac{1}{\mathrm{pow}_{\theta_0}(z,1/2)(z+1)} = \frac{\mathrm{pow}_{\theta_0}(z,-1/2)}{z+1}$. Then "starring" branch cut for this exercise will be $f_0(z) : \mathbb{C} \setminus \mathbb{R}_{\geqslant 0} \to \mathbb{C}$.

Let $\Gamma_{\epsilon,R,\theta}$ denote the contour $[\epsilon e^{i\theta} \to Re^{i\theta}] + \gamma_{0,R,\circlearrowleft}|_{[\theta,2\pi-\theta]} + [Re^{-i\theta} \to \epsilon e^{-i\theta}] + \gamma_{0,\epsilon,\circlearrowright}|_{[\theta,2\pi-\theta]}$ (which sort of looks like Pac-man with his mouth open, and closing his mouth as $\theta \searrow 0$). Note that $f_0(z)$ equals $f_{-\pi}(z) : \mathbb{C} \setminus \mathbb{R}_{\leqslant 0} \to \mathbb{C}$ on the straight line segment $[\epsilon e^{i\theta} \to Re^{i\theta}]$ for all small $\theta > 0$, and $f_0(z)$ equals $f_\pi(z) : \mathbb{C} \setminus \mathbb{R}_{\leqslant 0} \to \mathbb{C}$ on the straight line segment $[Re^{-i\theta} \to \epsilon e^{-i\theta}]$ for all small $\theta > 0$, where for $x \in (0,\infty)$, $f_{-\pi}(x) = \frac{\exp(-1/2\,\mathrm{Log}_{-\pi}(x))}{x+1} = \frac{\exp(-1/2\ln x)}{x+1} = \frac{1}{\sqrt{x}(x+1)}$ (recall that $\mathrm{Log}_{-\pi}$ is the "standard branch", which agrees with the real-analytic function $\ln x$ on $(0,\infty)$) and $f_\pi(x) = \frac{\exp(-1/2\,\mathrm{Log}_\pi(x))}{x+1} = \frac{\exp(-1/2(\ln x + 2\pi i))}{x+1} = \frac{\exp(-1/2\ln x)\exp(-\pi i)}{x+1} = -\frac{1}{\sqrt{x}(x+1)}$.

Then, using the "uniform convergence of paths $\implies$ convergence of integrals for continuous f" lemma I proved (using Riemann sums) in Ex. 37 of Homework 5, we have that

$$\int_{[\epsilon e^{i\theta} \to Re^{i\theta}]} f_0(z)\,dz = \int_{[\epsilon e^{i\theta} \to Re^{i\theta}]} f_{-\pi}(z)\,dz \to \int_{[\epsilon \to R]} f_{-\pi}(z)\,dz = \int_\epsilon^R \frac{1}{\sqrt{x}(x+1)}\,dx$$

as $\theta \searrow 0$; and similarly

$$\int_{[Re^{-i\theta} \to \epsilon e^{-i\theta}]} f_0(z)\,dz \to \int_{[R \to \epsilon]} f_\pi(z)\,dz = \int_R^\epsilon -\frac{1}{\sqrt{x}(x+1)}\,dx = \int_\epsilon^R \frac{1}{\sqrt{x}(x+1)}\,dx.$$

Returning to $\Gamma_{\epsilon,R,\theta}$, notice that this is a simple closed curve, whose interior contains the only singularity of $f_0(z)$ in $\mathbb{C} \setminus \mathbb{R}_{\geqslant 0}$, namely a simple pole at $z = -1$ (one could verify this rigorously by checking that the winding number of $\Gamma_{\epsilon,R,\theta}$ at -1 is non-zero, which is indeed doable since this contour is easily to explicitly integrate). The residue theorem then yields $\oint_{\Gamma_{\epsilon,R,\theta}} f_0(z)\,dz = 2\pi i \operatorname{Res}[f_0,-1]$. Using the residue formulas of Ex. 26 of Notes 4 (which we proved in last week's homework), $\operatorname{Res}[f_0,-1] = \operatorname{pow}_0(-1,-\frac{1}{2}) = \exp(-\frac{1}{2}\operatorname{Log}_0(-1)) = \exp(-\frac{1}{2}(\ln|1| + \operatorname{Arg}_0(-1))) = \exp(-\frac{1}{2}\pi) = -i$. We have the following bounds by the ML-estimate (note that $|f_0(z)| = \frac{|\exp(\frac{1}{2}\operatorname{Log}_0(z))^{-1}|}{|z+1|} \leqslant \frac{|\exp(\frac{1}{2}\operatorname{Log}_0(z))|^{-1}}{||z|-1|} = \frac{\exp(\mathfrak{Re}(\frac{1}{2}\operatorname{Log}_0(z)))}{||z|-1|} = \frac{\exp(\frac{1}{2}\mathfrak{Re}(\operatorname{Log}_0(z)))}{||z|-1|} = \frac{\exp(\frac{1}{2}\ln|z|)}{||z|-1|} = \frac{|z|^{-1/2}}{||z|-1|}$):

$$\left|\int_{\gamma_{0,R,\circlearrowleft}|_{[\theta,2\pi-\theta]}} f_0(z)\,dz\right| \leqslant 2\pi R \cdot \frac{R^{-1/2}}{R-1} \leqslant 2\pi R \cdot \frac{1}{R^{3/2}} = \frac{2\pi}{\sqrt{R}}$$

and

$$\left|\int_{\gamma_{0,\epsilon,\circlearrowleft}|_{[\theta,2\pi-\theta]}} f_0(z)\,dz\right| \leqslant 2\pi\epsilon \cdot \frac{\epsilon^{-1/2}}{1-\epsilon} \leqslant 2\pi\sqrt{\epsilon}.$$

Putting everything together (defining $I_{\epsilon,R} := \int_\epsilon^R \frac{1}{\sqrt{x}(x+1)}\,dx$), we have for all small $\theta > 0$,

$$\begin{aligned} |2\pi - 2I_{\epsilon,R}| &= \left|\oint_{\Gamma_{\epsilon,R,\theta}} f_0(z)\,dz - 2\int_\epsilon^R \frac{1}{\sqrt{x}(x+1)}\,dx\right| \\ &\leqslant \left|\int_{\gamma_{0,R,\circlearrowleft}|_{[\theta,2\pi-\theta]}} f_0(z)\,dz\right| + \left|\int_{\gamma_{0,\epsilon,\circlearrowleft}|_{[\theta,2\pi-\theta]}} f_0(z)\,dz\right| \\ &\quad + \left|\int_{[\epsilon e^{i\theta}\to Re^{i\theta}]} f_0(z)\,dz + \int_{[Re^{-i\theta}\to\epsilon e^{-i\theta}]} f_0(z)\,dz - 2I_{\epsilon,R}\right| \\ &\leqslant \frac{2\pi}{\sqrt{R}} + 2\pi\sqrt{\epsilon} + \left|\int_{[\epsilon e^{i\theta}\to Re^{i\theta}]} f_0(z)\,dz + \int_{[Re^{-i\theta}\to\epsilon e^{-i\theta}]} f_0(z)\,dz - 2I_{\epsilon,R}\right|. \end{aligned}$$

Taking the limit as $\theta \searrow 0$, we get $|2\pi - 2I_{\epsilon,R}| \leqslant \frac{2\pi}{\sqrt{R}} + 2\pi\sqrt{\epsilon}$, and taking the limit as $\epsilon \to 0, R \to \infty$ we get $2\int_0^\infty \frac{1}{\sqrt{x}(x+1)}\,dx = 2\pi \implies \int_0^\infty \frac{1}{\sqrt{x}(x+1)}\,dx = \pi$, and we are done.

Exercise 55

This exercise demonstrates that the hypothesis that U is simply connected used in the lifting lemma (Lemma 50 in Notes 4) is not *necessary* for the existence of a lift, by constructing an explicit example of a holomorphic function $f(z) : \mathbb{C}\setminus[0,1] \to \mathbb{C}$ s.t. $\exp(f(z)) = \frac{z-1}{z}$ (in other words for the non-simply connected set $U := \mathbb{C}\setminus[0,1]$, we are lifting the function $[z \to \frac{z-1}{z}] : U \to \mathbb{C}\setminus\{0\}$ to a function $f : U \to \mathbb{C}$ through the covering map $\exp(z) : \mathbb{C} \to \mathbb{C}\setminus\{0\}$).

Recall from the opening paragraphs of Section 4 in Notes 4 that we defined the "standard branch" of the (natural) logarithm, $\operatorname{Log}(z) = \ln|z| + i\operatorname{Arg}(z)$, on the slit plane $\mathbb{C}\setminus\mathbb{R}_{\leqslant 0}$, where $\operatorname{Arg}(z) : \mathbb{C}\setminus\{0\} \to \mathbb{C}$ is the unique $\theta \in (-\pi,\pi]$ s.t. $z = |z|e^{i\theta}$, and furthermore $\operatorname{Log}(z)$ is holomorphic on $\mathbb{C}\setminus\mathbb{R}_{\leqslant 0}$ and satisfies $\exp(\operatorname{Log}(z)) = z$ for $z \in \mathbb{C}\setminus\mathbb{R}_{\leqslant 0}$. Then, I claim that $f(z) := \operatorname{Log}(\frac{z-1}{z})$ is our desired holomorphic function $f(z) : \mathbb{C}\setminus[0,1] \to \mathbb{C}$ satisfying $\exp(f(z)) = \frac{z-1}{z}$.

It suffices to show that for any $z_0 \in \mathbb{C} \setminus [0,1]$, the complex number $\frac{z_0-1}{z_0}$ is in $\mathbb{C} \setminus \mathbb{R}_{\leqslant 0}$ — since then $\frac{z_0-1}{z_0} \in \mathbb{C} \setminus \mathbb{R}_{\leqslant 0} \implies \exp(f(z_0)) = \exp(\mathrm{Log}(\frac{z_0-1}{z_0})) = \frac{z_0-1}{z_0}$, and the chain rule gives that since the function $\frac{z-1}{z}$ is complex differentiable at $z_0 \in \mathbb{C} \setminus [0,1] \subseteq \mathbb{C} \setminus \{0\}$ and Log is complex differentiable at $\frac{z_0-1}{z_0} \in \mathbb{C} \setminus \mathbb{R}_{\leqslant 0}$, f must be complex differentiable at z_0. Well, suppose we are given $z_0 \in \mathbb{C} \setminus [0,1]$ s.t. $\frac{z_0-1}{z_0} = -a$ for some $a \in [0, \infty)$. Then, $z_0 - 1 = -az_0 \implies z_0 = \frac{1}{1+a} \in \mathbb{R}$, where $\frac{1}{1+a} \leqslant \frac{1}{1+0} = 1$ is obviously also $\geqslant 0$, i.e. $z_0 \in [0,1]$; contradiction.

Remark: the approach taken above works for this specific problem, but in the problem Terry hints at a perhaps more general method, namely by using the Schwarz reflection principle (which we proved in Ex. 37(OLD#)/39(NEW#) of Notes 3 in Homework 5) to extend a function defined on the (hopefully) simply connected open set $U \cap \overline{\mathbb{H}} := U \cap \{\mathtt{Im}(z) \geqslant 0\}$ to all of U (as long as the function is real on the real axis).

Exercise S&S §3.14

We want to prove that all entire functions that are also injective take the form $f(z) = az+b$ with $a, b \in \mathbb{C}$ and $a \neq 0$. As the hint suggests, we proceed by considering $f(\frac{1}{z})$ and applying Casorati-Weierstrass. First, we know (from our proof that holomorphic functions are analytic in Cor. 18(OLD#)/20(NEW#) of Notes 3) that the radius of convergence of the Taylor series of a holomorphic function $f : U \to \mathbb{C}$ at a point $z_0 \in U$ is $\geqslant$ any R s.t. $\overline{D(z_0, R)} \subseteq U$. In particular given an entire function f, we have that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for all $z \in \mathbb{C}$ and $a_n = f^{(n)}(z_0)$ (the Taylor series coefficients).

Consider now $g(z) := f(\frac{1}{z}) = \sum_{n=0}^{\infty} a_n z^{-n}$. If infinitely any a_n are non-zero, then g has an essential singularity at 0, so by Casorati-Weierstrass, $g(D(0,1) \setminus \{0\}) = f([|z| > 1])$ is dense in $\mathbb{C}$. By the open mapping theorem, $f(D(0,1))$ is an open (of course non-empty) set in $\mathbb{C}$, so in particular must intersect the dense set $f([|z| > 1])$ (at infinitely many points in fact, because a non-empty open set minus a point is still a non-empty open set), meaning in particular we have (at least) two distinct points $z \in D(0,1)$ and $z' \in [|z| > 1]$ s.t. $f(z) = f(z') \in f(D(0,1)) \cap f([|z| > 1])$, contradicting injectivity.

This rules out the case that infinitely many a_n are non-zero, so the remaining case is that f is a polynomial, say of degree N. Then if $N \geqslant 2$, I claim that f can not be injective (leaving of course the sole remaining possibility that $f(z) = az + b$, as desired — where of course $a \neq 0$, since otherwise f would be constant, obviously contradicting injectivity). By the fundamental theorem of algebra (FTA), f has exactly N roots (counting multiplicity), so if it is not the case that f has one root of multiplicity N, then multiple $z \in \mathbb{C}$ satisfy $f(z) = 0$, contradicting injectivity. Otherwise, we have $f(z) = (z - z_0)^N$. Then I claim that for any $0 \neq c \in \mathbb{C}$, $f(z) - c$ has at least 2 distinct roots, contradicting injectivity. This is because otherwise $f(z) - c = (z - z_1)^N$, and so comparing the coefficients of z^{N-1} we would have $-Nz_0 = -Nz_1 \implies z_0 = z_1$, but z_0 (which we supposed was the sole root of f!) can not be a root of $f(z) - c$ because $f(z_0) + c = 0 - c = -c \neq 0$.

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Daniel Rui - 11/19/21

Exercise 11 (Notes 4)

For an open connected set $U \subseteq \mathbb{C}$ and a meromorphic function $f : U \setminus S \to \mathbb{C}$, recall the following definition of $\operatorname{ord}_{z_0}(f)$ for the order of f at a point $z_0 \in U$:

- if f has a removable singularity at z_0 and has a zero of order m at z_0 once the singularity is removed, then $\operatorname{ord}_{z_0}(f) := m$;
- if f is holomorphic at z_0 and has a zero of order m at z_0 then $\operatorname{ord}_{z_0}(f) := m$;
- if f has a pole of order m at z_0 then $\operatorname{ord}_{z_0}(f) := -m$;
- if f is identically zero, then $\operatorname{ord}_{z_0}(f) = +\infty$.

These four bullet points can be summarized neatly as: $\operatorname{ord}_{z_0}(f)$ is the minimal $m \in \mathbb{Z}$ s.t. the Laurent series of f at z_0 has the coefficient $a_m \neq 0$ (where we take the convention that the minimum of an empty set is $+\infty$). We are then asked to establish the following facts (i.e. establish that order is a valuation):

(i) If $f_1 : U \setminus S_1 \to \mathbb{C}$ and $f_2 : U \setminus S_2 \to \mathbb{C}$ are equivalent meromorphic functions (which recall means that f_1, f_2 agree on their common domains of definition $U \setminus (S_1 \cup S_2)$), then $\operatorname{ord}_{z_0}(f) = \operatorname{ord}_{z_0}(g)$ for all $z_0 \in U$. In particular, this gives us that order is well-defined for functions in $\mathcal{M}(U)$, the set of meromorphic functions on U identified up to equivalence.

This is because the Laurent series (using uniqueness of Laurent series, Ex. 5 of Notes 4 which we did last week) of a meromorphic function F at z_0 is only defined by the values of F on $D(z_0, r) \setminus \{z_0\}$ for any small $r > 0$, and so because for any $z_0 \in U$ either $z_0 \in U \setminus (S_1 \cup S_2)$ in which case f, g are both defined and equal to each other on $D(z_0, r)$, or $z_0 \in S_1 \cup S_2$ is an isolated singularity (because both S_1, S_2 are closed discrete sets in U) in which case f, g are both defined and equal to each other on $D(z_0, r) \setminus \{z_0\}$, the Laurent series of f, g at z_0 are equal for all $z_0 \in U$.

(ii) If $f, g \in \mathcal{M}(U)$ and $z_0 \in U$, then $\operatorname{ord}_{z_0}(fg) = \operatorname{ord}_{z_0}(f) + \operatorname{ord}_{z_0}(g)$; and if g is not identically 0, then $\operatorname{ord}_{z_0}(\frac{f}{g}) = \operatorname{ord}_{z_0}(f) + \operatorname{ord}_{z_0}(g)$. This is because by the multiplication of power series (more details in Ex. 19 of Notes 1):

$$f \cdot g = \left(\sum_{n=\ell}^{\infty} b_n (z - z_0)^n \right) \cdot \left(\sum_{n=m}^{\infty} a_n (z - z_0)^n \right)$$
$$= b_\ell a_m (z - z_0)^{\ell + m} + b_\ell \cdot \left(\sum_{n=m+1}^{\infty} a_n (z - z_0)^{n+\ell} \right) + \left(\sum_{n=\ell+1}^{\infty} b_n (z - z_0)^n \right) \cdot \left(\sum_{n=m+1}^{\infty} a_n (z - z_0)^n \right)$$

and as the 2nd and 3rd terms in the above sum have exponents of $(z - z_0)$ strictly greater than $\ell + m$, and $b_\ell a_m \neq 0$ (since each individually is non-zero and $\mathbb{C}$ is a field so in particular an integral domain), so indeed $\ell + m = \operatorname{ord}_{z_0}(f) + \operatorname{ord}_{z_0}(g)$ is the minimal element m' of $\mathbb{Z}$ s.t. the coefficient $c_{m'}$ of fg is non-zero.

The division part follows from the multiplication part, and the fact that $\operatorname{ord}_{z_0}(\frac{1}{g}) = -\operatorname{ord}_{z_0}(g)$, which I justify here. For meromorphic $g = \sum_{n=m}^{\infty} a_n(z-z_0)^n \not\equiv 0$ with order $m \in \mathbb{Z}$ at z_0, we can run the procedure for dividing infinite series using the geometric series formula (we can first multiply by a constant $\lambda \in \mathbb{C}$ to make $a_m = 1$, and translate so that $z_0 = 0$):

$$\begin{aligned}\frac{1}{g(z)} = \lambda \cdot \frac{1}{\lambda g(z)} &= \frac{\lambda}{z^m} \cdot \frac{1}{1 - (\sum_{n=m+1}^{\infty} a_n z^{n-m})} \\ &= \frac{\lambda}{z^m} \cdot \left[1 + \left(\sum_{n=m+1}^{\infty} a_n z^{n-m}\right)^1 + \left(\sum_{n=m+1}^{\infty} a_n z^{n-m}\right)^2 + \dots\right]\end{aligned}$$

(more details found on pg. 153 of Gamelin) and so indeed $-m = -\operatorname{ord}_{z_0}(g)$ is the minimal element m' of $\mathbb{Z}$ s.t. the coefficient $c_{m'}$ of $\frac{1}{g}$ is non-zero.

(iii) If $f, g \in \mathcal{M}(U)$ and $z_0 \in U$, then $\operatorname{ord}_{z_0}(f+g) \geqslant \min\{\operatorname{ord}_{z_0}(f), \operatorname{ord}_{z_0}(g)\}$ (where we can sure that it is equality if $\operatorname{ord}_{z_0}(f) \neq \operatorname{ord}_{z_0}(g)$. Well if we have Laurent series $f(z) = \sum_{n=\ell}^{\infty} b_n(z-z_0)^n$ and $g(z) = \sum_{n=m}^{\infty} a_n(z-z_0)^n$, then for any $n < \min\{m, \ell\}$, their sum $\sum_{n=-\infty}^{\infty} c_n(z-z_0)^n$ will have $c_n = a_n + b_n = 0 + 0 = 0$, so indeed $\operatorname{ord}_{z_0}(f+g) \geqslant \min\{\operatorname{ord}_{z_0}(f), \operatorname{ord}_{z_0}(g)\}$. If $\ell = m$, then $c_m = b_\ell + a_m$, which might be zero thereby making the above inequality strict, but if say w.l.o.g. $m < \ell$, then $c_m = a_m \neq 0$ (and of course all $n < \min\{m, \ell\} = m$ would have $c_n = 0$), meaning that $\operatorname{ord}_{z_0}(f+g) = m = \min\{\operatorname{ord}_{z_0}(f), \operatorname{ord}_{z_0}(g)\}$, as desired.

Exercise 19

Let $U \subseteq \mathbb{C}$ be open and connected, and let S be a closed discrete subset of U with a function $f : U \setminus S \to \mathbb{C}$. We want to show that f is meromorphic on $U \iff f$ is the restriction of a holomorphic function $\tilde{f} : U \to \mathbb{C} \cup \{\infty\} =: \hat{\mathbb{C}}$ ($\hat{\mathbb{C}}$ denotes the Riemann sphere) that is not identically ∞, where furthermore if these equivalent conditions hold, $\tilde{f}$ is uniquely determined by f and is unaffected if f is replaced with an equivalent meromorphic function.

Before we begin, we should recall the following definitions: Def. 9 on Notes 4 says that f is meromorphic on $U \setminus S$ (where S, U have properties described above) if f is holomorphic on $U \setminus S$ and every $z_0 \in S$ is either a removable singularity or a pole of finite order; and the paragraph preceding Ex. 19 in Notes 4 says that $\tilde{f} : U \to \hat{\mathbb{C}}$ is holomorphic means precisely that $\tilde{f}$ is continuous, and f and $\frac{1}{f}$ (taking $\frac{1}{\infty} = 0$) are holomorphic respectively on the sets $\{z \in U : f(z) \neq \infty\}$ and $\{z \in U : f(z) \neq 0\}$ (which are open thanks to continuity of $\tilde{f}$).

($\implies$): for any $z_0 \in S$ that are removable singularities (say we split S into S_R and S_P for removable and pole singularities respectively), the Riemann removable singularity theorem tells us that we can "fill in the singularity" by continuity to get an extended function holomorphic locally at z_0, meaning that defining $\tilde{f}(z_0)$ to be that aforementioned "filled in value" we get that $\tilde{f}$ is holomorphic on $U \setminus S_P$. Then defining $\tilde{f}(z_0)$ for any $z_0 \in S_P$ to be ∞, then we get that $\tilde{f}$ is a continuous extension (by Thm. 12(ii) in Notes 4 which says that if z_0 is a pole of f then $|f(z)| \to \infty$ as $z \to z_0$).

Finally $\tilde{f}$ is also holomorphic locally at z_0 because by Ex. 11(ii) above we know that f having a pole of order $m \geqslant 1$ at z_0 means that $\frac{1}{f}$ has a zero of order m at z_0, so $\frac{1}{\tilde{f}}$ will be 0 at z_0 (using the convention $\frac{1}{\infty} = 0$) and equal to some power series $\sum_{n=m} a_n(z - z_0)^n$ for $m \geqslant 1$ near z_0, so indeed $\frac{1}{\tilde{f}}$ is locally equal to a power series (with non-negative exponents) at z_0 hence holomorphic locally at z_0, as desired.

($\Longleftarrow$): we are given that f is the restriction of a holomorphic function $\tilde{f} : U \to \hat{\mathbb{C}}$ that is not identically ∞. In particular it is continuous so the set of zeroes Z and the set of poles P are closed in U. Then $\tilde{f}$ restricted to $U \setminus P \to \mathbb{C}$ is a holomorphic function from an open subset of $\mathbb{C}$ to $\mathbb{C}$, so the identity theorem tells us that its set of zeroes, i.e. Z, is isolated (unless $\tilde{f}$ is identically 0 on $U \setminus P$ and hence on all of U by continuity, and the zero function 0 is obviously meromorphic on U). Similarly $\frac{1}{\tilde{f}}$ restricted to $U \setminus Z \to \mathbb{C}$ is a holomorphic function from an open subset of $\mathbb{C}$ to $\mathbb{C}$, so the identity theorem tells us that its set of zeroes, i.e. P, is isolated (we ruled out the case that this function could be identically 0 because we specified that $\tilde{f} \not\equiv \infty$).

Thus the restriction of $\tilde{f}$ to $U \setminus P$, denoted f, is a holomorphic function $U \setminus P \to \mathbb{C}$ with isolated singularities at $z_0 \in P$. We would be done if we knew that all such z_0 were removable singularities or poles, and indeed by continuity we know that $|f(z)| \to \infty$ as $z \to z_0$, and looking at Thm. 12 in Notes 4 we have that z_0 must in fact be a pole (poles are the only types of isolated singularities that have this behavior as $z \to z_0$), and we are done.

Finally the uniqueness claims. Since $\tilde{f} : U \to \hat{\mathbb{C}}$ is specified to be holomorphic, it must in particular continuous, and the values chosen above (filling in removable singularities and setting $\tilde{f}$ to be ∞ at poles z_0 or order $m \geqslant 1$) are indeed the continuous extensions (w.r.t. the topology of $\hat{\mathbb{C}}$) and so those are the only possible values that a continuous extension can possibly take. Lastly for two equivalent meromorphic functions f_1, f_2 at any $z_0 \in S_1 \cup S_2$, z_0 is an isolated singularity (because S_1, S_2 are closed and discrete) so they agree on any $D(z_0, r) \setminus \{z_0\}$ (I already discussed something similar in part (i) of Ex. 11 above) meaning any continuous extension like $\tilde{f}$ can only have one choice at z_0, and that choice is $\lim_{z \to z_0} f_1(z) = \lim_{z \to z_0} f_2(z)$ (limit taken in the $\hat{\mathbb{C}}$ topology).

Exercise 20

Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a holomorphic map from the Riemann sphere to itself. We want to show that f is either identically ∞, or is a rational function in the sense that there are polynomials $P(z), Q(z) : \mathbb{C} \to \mathbb{C}$ with $Q \not\equiv 0$ s.t. $f(z) = \frac{P(z)}{Q(z)}$ for all $z \in \mathbb{C}$ with $Q(z) \neq 0$. As the hint suggests, we proceed by showing that f has finitely many poles, which we then eliminate by multiplying with appropriate linear factors, and then apply Ex. 31 from Notes 3 (a corollary of the ML-estimate on the Cauchy integral formulas for the Taylor series coefficients, that says that if there exist $M > 0$ and $A \geqslant 0$ s.t. $|f(z)| \leqslant M(1+|z|)^A$ for all $z \in \mathbb{C}$ then f is in fact a polynomial). Let S denote the set of $z \in \mathbb{C}$ s.t. $f(z) = \infty$. I argued above in the ($\Longleftarrow$) direction Ex. 19 using the identity theorem that S must be closed and isolated (unless f is identically ∞). In particular this means that the number of poles in any compact set must be finite.

We set out now to refine this into the statement that S in all of $\mathbb{C}$ is finite. Observe that the map $[z \mapsto \frac{1}{z}] : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a holomorphic function of the Riemann sphere (recall from the paragraph preceding Ex. 19 in Notes 4 that $g : U \to \hat{\mathbb{C}}$ (for open $U \subseteq \hat{\mathbb{C}}$) holomorphic precisely means that $[z \mapsto g(z)]$ is holomorphic on $U \cap \mathbb{C}$ and $[z \mapsto g(\frac{1}{z})]$ is holomorphic on $\{z \in \mathbb{C} : \frac{1}{z} \in U\}$; in our case with $g(z) = \frac{1}{z}$ and $U = \hat{\mathbb{C}}$ this is "$[z \mapsto \frac{1}{z}]$ is holomorphic on $\mathbb{C}$ and $[z \mapsto z]$ is holomorphic on $\mathbb{C}$", which is true. The composition of holomorphic maps on Riemann surfaces is also a holomorphic map, so $h(z) := f(\frac{1}{z})$ is a holomorphic map.

In particular we have that $h(z)$ has finitely many poles on the compact set $\overline{D(0,1)}$, which means that $f(z)$ has finitely many poles on the set $\hat{\mathbb{C}} \setminus D(0,1)$. As $f(z)$ of course has finitely many poles on the compact set $\overline{D(0,1)}$, we get that f has in total finitely many poles in $\mathbb{C}$. Intuitively, this whole "inversion $\frac{1}{z}$ business" is basically saying that having infinitely many poles would lead to an "accumulation" at $z = \infty$, and the set of poles can not have an accumulation point (unless $f \equiv \infty$ of course) — we went to all this trouble because our tools like the identity theorem we only proved for open $U \subseteq \mathbb{C}$. Of course there might be a cleverer way to use existing tools but I am not "paid to be clever".

Let $Q(z) := \prod_{z_0 \in S} (z - z_0)^{-\operatorname{ord}_{z_0}(f)}$ (finite product of linear factors) so that the order of $f(z)Q(z)$ at $z_0 \in S$ (using Ex. 11(ii) above) is 0. Using the Riemann removable singularity theorem we may fill in all these singularities and get a holomorphic extension $F : \mathbb{C} \to \mathbb{C}$ of $f(z)Q(z)$. We may even extend it to a holomorphic function on $\hat{\mathbb{C}}$ since f, Q are both holomorphic functions on $\hat{\mathbb{C}}$. If $F(\infty) \neq \infty$, then F will be bounded on a neighborhood of ∞, i.e. $D(z_0, R)^{\mathsf{C}}$, and of course it will be bounded on $\overline{D(z_0, R)}$ by continuity, so by Liouville's theorem F will be constant on $\mathbb{C}$ and indeed $F(z) = \frac{1}{Q(z)}$ on $\mathbb{C}$ (where $Q(z) \neq 0$) as desired.

Otherwise $F(\infty) = \infty$. If we can say that F has a "pole of order m at ∞", then intuitively we would get something like $|\frac{f(z)}{z^m}|$ bounded on a neighborhood of ∞, i.e. $D(z_0, R)^{\mathsf{C}}$. And then we'd be done by appealing to Ex. 31 as we mentioned before in the 1st paragraph of this problem (since we can bound this by M on $D(z_0, R)^{\mathsf{C}}$ and f by M on $\overline{D(z_0, R)}$ meaning $|f(z)| \leqslant M + M|z|^m \leqslant M(1 + |z|)^m$, using say the binomial theorem for that last inequality).

We make this rigorous as follows: consider $H(z) := F(\frac{1}{z})$ (which is holomorphic map $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ as we discussed for $h(z) := f(\frac{1}{z})$ several paragraphs ago). Ex. 19 above gives that H is meromorphic on $D(0, r)$, and by the assumption $F(\infty) = \infty$ we know that $H(0) = \infty$ so H has a pole of finite order, say m, at 0. Then $|H(z)z^m|$ approaches some finite value as $z \to 0$ (in fact the absolute value of the $(-m)$th coefficient of the Laurent series), so $|H(z)z^m|$ is bounded by some M on $D(0, \frac{1}{R})$. Inverting everything we get $|\frac{F(z)}{z^m}|$ is bounded by M on $D(0, R)^{\mathsf{C}}$, and everything goes through the way we said it would in the previous paragraph.

Exercise 26

We wish to prove the following residue formulas:

(i) if f has a removable singularity at z_0 then $\text{Res}[f, z_0] = 0$;

(ii) if f has a simple pole at z_0, then $\text{Res}[f, z_0] = \lim_{z\to z_0} f(z)(z - z_0)$ where this limit exists and is a complex number, i.e. not ∞ or something (the limit is taken with $z \notin U \setminus S$ and $z \neq z_0$);

(iii) and more generally if f has a pole of order $\leqslant m$ at z_0 for some $m \geqslant 1$, then $\text{Res}[f, z_0] = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}[f(z)(z - z_0)^m]$); in particular if $f(z) = \frac{g(z)}{(z-z_0)^m}$ near z_0 for some g holomorphic near z_0, then $\text{Res}[f, z_0] = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} g(z_0)$.

As residues are defined as the a_{-1} coefficient in the Laurent series at z_0, the proofs of these formulas hinge upon manipulations with said Laurent series. For (i), the Riemann removable singularity theorem (Ex. 35 of Notes 3, which we did in Homework 5) tells us that f can be extended to a holomorphic function $\tilde{f}$ near z_0, and as holomorphic functions are analytic, $\tilde{f}$ admits a Taylor series at z_0, and f agrees with $\tilde{f}$ on $D(z_0, r) \setminus \{z_0\}$ and Laurent series are unique, we have that the Laurent series of f at z_0 is in fact the Taylor series of $\tilde{f}$ at z_0, which of course has $a_{-1} = 0$.

Next I prove (iii), which of course has (ii) as a special case. The assumption that f has a pole of order $\leqslant m$ at z_0 means that the Laurent series of f can be written as $\sum_{n=-m}^{\infty} a_n(z - z_0)^n$, and so $f(z)(z - z_0)^m$ is a power series with no negative-exponent terms, namely $\sum_{n=0}^{\infty} a_{n-m}(z - z_0)^n$. Let us define $b_n := a_{n-m}$ so that this power series can be written as $\sum_{n=0}^{\infty} b_n(z-z_0)^n$. From Thm. 15 of Notes 1 we know that the derivative of this power series is given by term-by-term differentiation, and indeed the $(m-1)$th derivative is $\frac{d^{m-1}}{dz^{m-1}}[f(z)(z-z_0)^m]$ equals $S(z) := \sum_{n=0}^{\infty}(n+(m-1)) \cdots (n+1)b_{n+(m-1)}(z-z_0)^n$ (perhaps not at z_0 since f may not be defined there, but certainly on a small punctured disk $D(z_0, r) \setminus \{z_0\}$). Finally, note that $\text{Res}[f, z_0] := a_{-1} = b_{m-1}$ is the constant term in the power series $S(z)$, i.e. $S(z_0)$, and as Thm. 15 of Notes 1 tells us functions defined by power series are holomorphic, S is in particular continuous so $\text{Res}[f, z_0] = S(z_0) = \lim_{z\to z_0} S(z) = \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}[f(z)(z - z_0)^m]$ (where the limit is taken for $z \in D(z_0, r)$), as desired.

Our last task is to use these residue formulas to show that Cauchy's theorem (for simply connected U), the general Cauchy integral formula (for simply connected U), and the higher-order general Cauchy integral formulas (for simply connected U), i.e. Thm. 14, Thm. 39, Ex. 40 of Notes 3 in the OLD — pre 11/9/21 — numbering; or Thm. 14, Thm. 41, Ex. 42 of Notes 3 in the NEW — post 11/9/21 — numbering, are all subsumed by the residue theorem. Let us state the residue theorem (Thm. 22 of Notes 4): for a simply connected open set $U \subseteq \mathbb{C}$, closed discrete singular set S, holomorphic $f : U \setminus S \to \mathbb{C}$, and a closed curve γ in $U \setminus S$, we have that $\oint_\gamma f(z)\,dz = 2\pi i \sum_{z_0\in S} W_\gamma(z_0) \text{Res}[f, z_0]$.

- For Cauchy's theorem, we have open $U \subseteq \mathbb{C}$ simply connected, a curve γ in U, and holomorphic $f : U \to \mathbb{C}$, so the singular set is empty and the residue theorem gives us $\oint_\gamma f(z)\,dz = \sum_{z_0\in\varnothing} = 0$.
- For the general Cauchy integral formula, we have open $U \subseteq \mathbb{C}$ simply connected, a curve γ in U avoiding z_0, and holomorphic $f : U \to \mathbb{C} \implies \frac{f(z)}{z-z_0}$ is holomorphic on $U \setminus S$ for $S := \{z_0\}$; then $\oint_\gamma f(z)\,dz = 2\pi i \sum_{z_0\in S} W_\gamma(z_0) \text{Res}[f, z_0] = 2\pi i W_\gamma(z_0) \text{Res}[f, z_0]$. By the "in particular" portion of (iii) above, taking $m = 1$, we get $\text{Res}[f, z_0] = f(z_0)$, and so indeed $\oint_\gamma f(z)\,dz = 2\pi i W_\gamma(z_0)f(z_0)$.
- For the higher-order general Cauchy integral formula, we have open $U \subseteq \mathbb{C}$ simply connected, a

curve γ in U avoiding z_0, and holomorphic $f : U \to \mathbb{C} \implies \frac{f(z)}{(z-z_0)^{n+1}}$ is holomorphic on $U \setminus S$ for $S := \{z_0\}$; then $\oint_\gamma f(z)\,dz = 2\pi i W_\gamma(z_0)\operatorname{Res}[f, z_0]$. By the "in particular" portion of (iii) above, taking $m = n + 1$, we get $\operatorname{Res}[f, z_0] = \frac{1}{n!} f^{(n)}(z_0)$, and so indeed $n! \oint_\gamma f(z)\,dz = 2\pi i W_\gamma(z_0) f^{(n)}(z_0)$.

Exercise 27

We are asked to use the residue theorem to give another proof of the fundamental theorem of algebra, by considering the integral $\oint_{\gamma_{0,R,\circlearrowleft}} \frac{z^{n-1}}{P(z)}\,dz$ for a polynomial $P(z) := a_n z^n + \ldots + a_0$ of degree $n \geqslant 1$ and large $R > 0$ (the factor theorem, which holds in any field, tells us P has $\leqslant n$ roots, so all large enough R will avoid the set $S = V(P)$ of roots of $P \equiv$ the poles of $\frac{z^{n-1}}{P(z)}$), and showing that it is non-zero; as the residue theorem (Cor. 23 of Notes 4) tells us that this integral is also equal to $\sum_{z_0 \in S} \operatorname{Res}[f, z_0]$, the integral being non-zero means that some residue $\operatorname{Res}[f, z_0]$ is non-zero, and so $f(z) := \frac{z^{n-1}}{P(z)}$ must have a pole at $z_0 \implies P(z)$ has a zero at z_0. And of course as in the proof of the FTA in Thm. 32 of Notes 3, induction and the factor theorem gets us to exactly n roots (counting multiplicity).

Let us assume w.l.o.g. that $a_n = 1$ (otherwise we can divide by $a_n \neq 0$ without changing whether or not P has zeroes in $\mathbb{C}$). Then,

$$\left|f(z) - \tfrac{1}{z}\right| = \left|\frac{z^n - P(z)}{zP(z)}\right| \leqslant \left|\frac{1}{z}\right| \cdot \frac{|a_{n-1}z^{n-1}| + \ldots + |a_0|}{|z^n| - |a_{n-1}z^{n-1} + \ldots + a_0|}.$$

Because $\frac{|z|^j}{|z|^n} \to 0$ for $j < n$, there is some $R_0 > 0$ s.t. $|z| > R_0 \implies |a_{n-1}z^{n-1}| + \ldots + |a_0| \leqslant (|a_{n-1}| + 1)|z^{n-1}|$, and $|z| > R_0 \implies |z^n| - |a_{n-1}z^{n-1} + \ldots + a_0| \geqslant \frac{1}{2}|z^n|$. Then we get $\left|f(z) - \frac{1}{z}\right| \leqslant |\frac{1}{z}| \cdot \frac{(|a_{n-1}|+1)}{(1/2)|z|}$ which is $\leqslant C|\frac{1}{z^2}|$ for some $C > 0$ (namely $C := 2(|a_{n-1}|+1)$). By the ML-estimate we get (for R larger than the absolute values of the $\leqslant n$ roots of P, and larger than R_0) $\oint_{\gamma_{0,R,\circlearrowleft}} |f(z) - \frac{1}{z}|\,dz \leqslant 2\pi R \cdot \frac{C}{R^2} = \frac{2\pi C}{R}$ which goes to 0 as $R \to \infty$. Thus we get that for large enough $R > 0$,

$$\left|\oint_{\gamma_{0,R,\circlearrowleft}} f(z)\,dz - 2\pi i\right| = \left|\oint_{\gamma_{0,R,\circlearrowleft}} f(z)\,dz - \oint_{\gamma_{0,R,\circlearrowleft}} \frac{1}{z}\,dz\right| \leqslant \oint_{\gamma_{0,R,\circlearrowleft}} \left|f(z) - \frac{1}{z}\right| dz \leqslant \frac{2\pi C}{R},$$

so the integral $\oint_{\gamma_{0,R,\circlearrowleft}} f(z)\,dz$ for all large enough $R > 0$ is non-zero (because RHS gets arbitrarily small).

Remarks: note that Cauchy's theorem could have worked just as well instead of the residue theorem, since the integral being non-zero means that $f(z)$ can't have been holomorphic on all of $\mathbb{C}$, implying that $P(z)$ must have had some zero in $\mathbb{C}$. However, an interesting consequence of the residue theorem is that because for all large enough $R > 0$, the integral $\oint_{\gamma_{0,R,\circlearrowleft}} f(z)\,dz = \sum_{z_0 \in V(P)} \operatorname{Res}[f, z_0]$ does not change with R, so $\oint_{\gamma_{0,R,\circlearrowleft}} [f(z) - \frac{1}{z}]\,dz = \oint_{\gamma_{0,R,\circlearrowleft}} f(z)\,dz - 2\pi i$ also does not change with R, so indeed for all large enough $R > 0$, $\oint_{\gamma_{0,R,\circlearrowleft}} \frac{z^n - P(z)}{zP(z)}\,dz$ is exactly 0!

Further remarks: the residue theorem can also be used in different ways to prove the FTA — one method is to consider $\int_{\gamma_{0,R,\circlearrowleft}} \frac{1}{zP(z)}\,dz$ (this proof, like the one we did in this exercise, is essentially a corollary of Cauchy's theorem, not needing anything remotely close to the full power of the residue theorem), and another is to use the argument principle, i.e. consider $\int_{\gamma_{0,R,\circlearrowleft}} \frac{P'(z)}{P(z)}\,dz$.

246A Homework 6

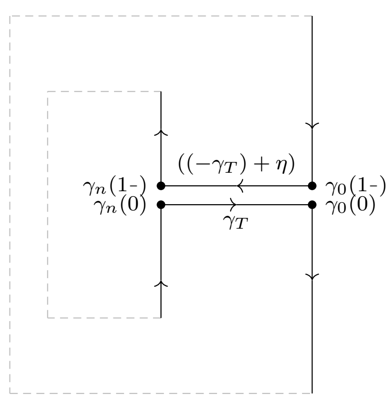
Daniel Rui - 11/12/21

Exercise 3 (Notes 4)

Let us first review Ex. 2 of Notes 4 (the impatient reader may skip to the problem below starting at the second big blue *Theorem* box):

> *Theorem: Cauchy integral formula on generalized annuli*
>
> Let $\gamma_0, \gamma_1, \ldots, \gamma_n$ be a simple closed curves oriented counterclockwise (which recall is defined using winding numbers for points in $\text{int}(\gamma_0)$), where the images and interiors of $\gamma_1, \ldots, \gamma_n$ are disjoint and all lie in $\text{int}(\gamma_0)$. Let $U \subseteq \mathbb{C}$ be an open set containing $\gamma_0, \ldots, \gamma_n$ and the region $\Omega := \text{int}(\gamma_0) \cap \text{ext}(\gamma_1) \cap \ldots \cap \text{ext}(\gamma_n)$, and let $f : U \to \mathbb{C}$ be a holomorphic functions. Then, for any $z_0 \in \Omega$,
> $$f(z_0) = \frac{1}{2\pi i} \oint_{\gamma_0} \frac{f(z)}{z - z_0}\, dz - \sum_{j=1}^{n} \frac{1}{2\pi i} \oint_{\gamma_j} \frac{f(z)}{z - z_0}\, dz.$$

We are not asked to prove it, but I will give a quick sketch nonetheless. Lemma 1 of Notes 4 proves the claim for $n = 1$ (see proof in Notes 4 or my thoughts on it in my bright blue 2021 notebook). We used two major "techniques" we have seen before. (1) in the final part of Terry's proof on JCT in Notes 3, he proves that any simple closed curve can be "approximated" by a polygonal path $\gamma_{\urcorner}$ with only vertical and horizontal lines (as denoted in the symbol $\gamma_{\urcorner}$), in the sense that for any ϵ-neighborhood of $\text{im}(\gamma)$ we can find a vertical-horizontal polygon $\gamma_{\urcorner}$ lying in that ϵ-neighborhood intersected with $\text{int}(\gamma)$; this allows us (fixing $z_0 \in \Omega$ at the beginning) to reduce all $\gamma_0, \gamma_1, \ldots, \gamma_n$ to be vertical-horizontal polygons without impacting the integrals $\oint \frac{f(z)}{z-z_0}\, dz$ (using Cauchy on the homotopy result now known as Prop. 62 in Notes 3 — used to be Prop. 57). And (2) at the end of Terry's proof of Lemma 1 in Notes 4 (or rather in my recounting of Terry's proof in my bright blue 2021 notebook) we "tunnelled" via a "tunnel path" γ_T from a vertical edge of γ_n to a vertical edge γ_0 (we can avoid vertices of γ_0 by small perturbations) to create a simple closed curve $\Gamma_{1\text{-}} := \gamma_0|_{[0,1\text{-}]} + ((-\gamma_T) + \eta) + (-(\gamma_n|_{[0,1\text{-}]})) + \gamma_T$, pictured below:



Just a reminder, we used the above (2) to say that for any function F holomorphic on $U \supseteq \Omega$ (which recall in Lemma 1 was just $\Omega := \text{int}(\gamma_0) \cap \text{ext}(\gamma_1)$) satisfies $\oint_{\gamma_0} F(z)\,dz - \oint_{\gamma_n} F(z)\,dz = \int_{\Gamma_1} F(z)\,dz = \lim_{1\text{-}\nearrow 1} \oint_{\Gamma_{1\text{-}}} F(z)\,dz = 0$ (more details about why we can apply Cauchy to $\oint_{\Gamma_{1\text{-}}} F(z)\,dz$ to get 0 are again on the final pages of my bright blue 2021 notebook); as for why we'd want this result about holomorphic F on $U \supseteq \Omega$, reread Terry's proof of Lemma 1. Anyways, for the general result for $n \in \mathbb{N}$, we use induction, and for the induction step, we use both (1) and (2) above to reduce from n to $(n-1)$ "holes" in the "annulus" in either case of the horizontal tunnel from γ_n (A) hitting γ_0 first, yielding the "holes" $\gamma_1, \ldots, \gamma_{n-1}$ in the "new γ_0" $\Gamma_{1\text{-}}$; or (B) hitting some other γ_i first, in which case we can rename every curve so that γ_n is now the last curve in $\{\gamma_1, \ldots, \gamma_n\}$ hit by the tunnel before the tunnel hits γ_0, thereby reducing to case (A). Finally taking limits (using uniform convergence of the paths $\Gamma_{1\text{-}} \rightrightarrows \Gamma_1$; see lemma I proved last week for Ex. 37) we have $\oint_{\gamma_0} \frac{f(z)}{z-z_0}\,dz - \oint_{\gamma_n} \frac{f(z)}{z-z_0}\,dz = \int_{\Gamma_1} \frac{f(z)}{z-z_0}\,dz = \lim_{1\text{-}\nearrow 1} \oint_{\Gamma_{1\text{-}}} \frac{f(z)}{z-z_0}\,dz$, and since the induction hypothesis for $(n-1)$ "holes" in $\text{int}(\Gamma_{1\text{-}})$ (for *any* 1- close enough to 1) gives that $f(z_0) = \frac{1}{2\pi i} \oint_{\Gamma_{1\text{-}}} \frac{f(z)}{z-z_0}\,dz - \sum_{j=1}^{n-1} \frac{1}{2\pi i} \oint_{\gamma_j} \frac{f(z)}{z-z_0}\,dz$, plugging in $\oint_{\Gamma_{1\text{-}}} \frac{f(z)}{z-z_0}\,dz = \oint_{\gamma_0} \frac{f(z)}{z-z_0}\,dz - \oint_{\gamma_n} \frac{f(z)}{z-z_0}\,dz$ yields the result.

Remark: above we used but did not discuss that the images and interiors of $\gamma_1, \ldots, \gamma_{n-1}$ still lie in $\text{int}(\Gamma_{1\text{-}})$; this is due to the fact from JCT that interiors of curves can be defined in terms of the winding number, and as $\Gamma_{1\text{-}} \rightrightarrows \Gamma_1$ (hence the integrals defining the winding numbers must converge) and winding numbers have to be integers, the images and interiors of $\gamma_1, \ldots, \gamma_{n-1}$ are in the interior of $\Gamma_{1\text{-}}$ for 1- close enough to 1. I used this technique in my breakdown of Terry's proof of Lemma 1, again in the final pages of my bright blue 2021 notebook.

The Actual Problem

We are ready for Ex. 3 now (a significant generalization to Riemann's theorem on removable singularities, which we proved last week in Ex. 35; and as is remarked in the problem, further inquiry leads to the study of analytic capacity):

> *Theorem: Painlevé's theorem on removable singularities*
>
> Let $U \subseteq \mathbb{C}$ be open, and let S be a compact subset of U which has zero length in the sense that for any $\epsilon > 0$, S can be covered by a countable number of disks $D(z_n, r_n)$ s.t. $\sum_{n=1}^{\infty} r_n < \epsilon$. Let $f : U \setminus S \to \mathbb{C}$ be a bounded holomorphic function. Then, the singularities of S are removable in the sense that there is a holomorphic extension $\tilde{f} : U \to \mathbb{C}$ of f to all of U.

Proof: as the provided hint suggests, we will use Ex. 2, which we discussed in some detail above. Fix $\epsilon > 0$ arbitrarily small, and let $\{D(z_n, r_n)\}_{n=1}^{\infty}$ be a cover of S with "length" $\sum_{n=1}^{\infty} r_n < \epsilon$. As S is compact, there is a finite subcover $\{D(z_n, r_n)\}_{n=1}^{N}$ that covers S. As S is compact and disjoint compact/closed sets are distant, $d(S, U^{\complement}) > 0$, so for all small enough ϵ, the ϵ-neighborhood of S (which contains all the $D(z_n, r_n)$ since $D(z_n, r_n)$ covers some point of S and has radius $r_n < \sum_{n=1}^{\infty} r_n < \epsilon$) lies completely in U. Let $D(z_n, r_n^+)$ be a slightly expanded version of $D(z_n, r_n)$ that still keeps the "length" $< \epsilon$ (hence still lies within ϵ-neighborhood of S, which is within U), but has the added benefit that $\overline{D(z_n, r_n)} \subseteq D(z_n, r_n^+)$.

Let B be any open ball s.t. $\overline{B} \subseteq U$, and observe that if we prove the claim for B, i.e. prove that f holomorphic on $B \setminus S$ can be extended to $\tilde{f}_B$ holomorphic on all of B, then we can "glue" all these extensions together into a "collage" function $\tilde{f}$ defined on all of U (since $U = \bigcup_{\overline{B} \subseteq U} B$) that is well-defined, since any two $\tilde{f}_B$ and $\tilde{f}_{B'}$ agree on the (perhaps empty) open set $(U \setminus S) \cap B \cap B'$, and furthermore holomorphic on U since holomorphicity is local (i.e. as long as $\tilde{f}_B$ is holomorphic at some $p \in U$, the "collage" function $\tilde{f}$ will be as well). So let us fix such a B, and also some $z_0 \in B \setminus S$.

Let γ_0 be the (counterclockwise) circular path traversing boundary $\partial B \subseteq U$ (say using the standard parameterization). **Caveat:** we will later be integrating a function involving f over this curve γ_0, so there is a problem if γ_0 hits some point of S. In this case, let the "new γ_0" be the "outer edge" of the curve γ_0 and all the circular boundaries $\partial D(z_n, r_n^+)$ for all $D(z_n, r_n^+)$ in the connected component of $B \cup \bigcup_{n=1}^{N} D(z_n, r_n^+)$ that contains B (denote the set of such $n \in [N]$ by N_B).

I mean the phrase "outer edge" as follows: the complement of the union of the above curves $\text{im}(\gamma_0) \cup \bigcup_{n \in N_B} \partial D(z_n, r_n^+)$ is an open set, which we can split into connected components; then there is only one unbounded component Ω_∞ (same reason as why there is only one unbounded component in the JCT), and the boundary of this unbounded component is a simple closed curve $\tilde{\gamma}_0$[†]; and furthermore $\text{int}(\tilde{\gamma}_0)$ contains $B \cup \bigcup_{n \in N_B} D(z_n, r_n^+)$ because otherwise those points would have to be in the exterior $\text{ext}(\tilde{\gamma}_0) = \Omega_\delta$[*] (which consists of points path connected "to ∞", i.e. a very far away point, where the path does not hit any point of $\text{im}(\gamma_0) \cup \bigcup_{n \in N_B} \partial D(z_n, r_n^+)$), but that's impossible since any path "to ∞" starting from one of those points would hit γ_0 or $\partial D(z_n, r_n^+)$. Terry makes a similar argument (at the end of his proof of the JCT) about the "inner edge" of a covering of the Jordan curve γ by squares being a simple closed path lying within an ϵ-neighborhood of $\text{im}(\gamma)$.

We show that any $z_0 \in B \setminus S \subseteq \text{int}(\gamma_0)$, $f(z_0) = \frac{1}{2\pi i} \int_{\gamma_0} \frac{f(z)}{z - z_0}\, dz$, thereby showing the desired claim for B because the function on the RHS, $\tilde{f}_B(w) := \frac{1}{2\pi i} \int_{\gamma_0} \frac{f(z)}{z-w}\, dz$ is holomorphic on all of B by Ex. 38 of Notes 3 (the exact same claim regarding the holomorphicity of this integral expression appears in Terry's proof of Lemma 1, and he also appeals to Ex. 38 of Notes 3). More explicitly, Ex. 38 of Notes 3 says that if for every fixed $t \in [0,1]$ the function $[w \mapsto F(t,z)]$ is holomorphic on an open set V, then $[w \mapsto \int_0^1 F(t,w)\, dt]$ is holomorphic on V; here, since γ_0 is piecewise $\mathcal{C}^1$ we can use Ex. 17(iv) of Notes 2 to transform the "dz" integral into a "dt" integral; and for every fixed t, corresponding to fixed point $z \in \text{im}(\gamma_0)$, the function $[w \mapsto \frac{f(z)}{z-w}]$ is holomorphic on $\mathbb{C} \setminus \text{im}(\gamma_0) \supseteq B$, and so Ex. 38 applies.

[†] see 11/12-13/21 thoughts in (hard-cover) gray notebook II for more rigorous discussion of this (I also note that it is easier to do everything with squares instead of circles; later in the proof when "zero length" is used in the ML-estimate, squares and circles work equally well). I proved only that $\text{im}(\tilde{\gamma}_0)$ is a union of disjoint simple closed curves (I mentioned in the notebook that disjointness might require a small perturbation, namely increasing the radii a tiny bit — keeping "length" $< \epsilon$ — to make two almost-connected components into one connected component), but the fact that there's only one I think comes from connectedness of the the component $B \cup \bigcup_{n=1}^{N} D(z_n, r_n^+)$. I know in Terry's JCT square-covering proof he also argues at one point that "inner edge" consists of only one simple closed curve due to connectedness of the union of the interiors of the squares.

[*] The ($\supseteq$) direction is because Ω_δ consists of all points path connected to ∞ with path not hitting $\text{im}(\gamma_0) \cup \bigcup_{n \in N_B} \partial D(z_n, r_n^+)$, which is subset of $\text{ext}(\tilde{\gamma}_0)$, i.e. all points path connected to ∞ with path not hitting $\text{im}(\tilde{\gamma}_0) \subseteq \text{im}(\gamma_0) \cup \bigcup_{n \in N_B} \partial D(z_n, r_n)$. The ($\subseteq$) direction is because the interior, image, and exterior of $\tilde{\gamma}_0$ partition the plane, and as Ω_∞ does not intersect with $\text{im}(\tilde{\gamma}_0)$, if it did intersect with $\text{int}(\tilde{\gamma}_0)$ that would contradict the (path-)connectedness of Ω_∞ since we know for sure it intersects $\text{ext}(\tilde{\gamma}_0)$.

Ok, we are finally at the stage to use Ex. 2 above and finish the problem. From everything above, we have a counterclockwise simple closed curve $\gamma_0 \subseteq U \setminus S$ (we put a lot of work in to ensure it does not hit a point of S!) that contains $\bigcup_{n=1}^N \overline{D(z_n, r_n)} \subseteq B \cup \bigcup_{n=1}^N D(z_n, r_n^+)$ in its interior $\mathrm{int}(\gamma_0)$ (this is another benefit of the "outer edge" construction of γ_0 we used, since the original γ_0 defined as ∂B might not completely contain $\overline{D(z_n, r_n)}$ for $D(z_n, r_n)$ that intersect with B).

Forming connected components of the $D(z_n, r_n)$ (for $n \in N_B$) and performing the "outer edge" construction discussed above, we get finitely many (say K) disjoint simple closed curves that lie in $\bigcup_{n\in N_B} \partial D(z_n, r_n) \subseteq B \cup \bigcup_{n=1}^N D(z_n, r_n^+) \subseteq \mathrm{int}(\gamma_0)^\dagger$. If we have that $\mathrm{im}(\gamma_j) \subseteq \mathrm{int}(\gamma_k)$ for some $j \neq k \in \{1, \dots, K\}$ (which can happen; see gray notebook II 11/14/21 for illustration), then we can discard γ_j (it is enough that the singularities enclosed by γ_j are enclosed by γ_k). This allows us to reduce to the case where all the γ_j for $j \in \{1, \dots, K\}$ avoid each others interiors. We are now ready to apply Ex. 2, the statement of which recall is in the first big blue *Theorem* box above.

By construction of the γ_j in the previous paragraph (we can ask that they be oriented counterclockwise), all the singularities $S \cap \mathrm{int}(\gamma_0)$ are enclosed within some γ_j for $j \in \{1, \dots, K\}$, so f is holomorphic on $\Omega := \mathrm{int}(\gamma_0) \cap \mathrm{ext}(\gamma_1) \cap \dots \cap \mathrm{ext}(\gamma_K) \subseteq U \setminus S$. Ex. 2 gives that for any $z_0 \in \Omega$,

$$\left| f(z_0) - \tfrac{1}{2\pi i} \oint_{\gamma_0} \tfrac{f(z)}{z - z_0}\, dz \right| = \left| \sum_{j=1}^K \tfrac{1}{2\pi i} \oint_{\gamma_j} \tfrac{f(z)}{z-z_0}\, dz \right| \leqslant \tfrac{1}{2\pi} \sum_{j=1}^K \left| \oint_{\gamma_j} \tfrac{f(z)}{z-z_0}\, dz \right|.$$

By assumption f is bounded on $U \setminus S$ (by say $M > 0$), and when we fixed $z_0 \in B \setminus S$ we see that all sufficiently small $\epsilon > 0$ are s.t. $\overline{D(z_n, r_n)}$ (a cover depending on ϵ) being in an ϵ-neighborhood of S means that $d\big(z_0, \bigcup_{n=1}^N \overline{D(z_n, r_n)}\big) \geqslant \Delta > 0$ for some positive distance $\Delta > 0$ (again using that disjoint compact/closed sets are distant). That is to say fixing $\epsilon > 0$ small enough at the beginning ensures that $\mathrm{im}(\gamma_j) \subseteq \bigcup_{n \in N_B} \partial D(z_n, r_n) \subseteq \bigcup_{n=1}^N \overline{D(z_n, r_n)}$ is a distance of $\geqslant \Delta > 0$ from z_0. Thus for all the integrals over γ_j on the RHS above, the function $\frac{f(z)}{z-z_0}$ is bounded by $\frac{M}{\Delta}$. The sum of the lengths of the γ_j is bounded by the sum Σ of the lengths of $\partial D(z_n, r_n)$ over $n \in [N]$ (because the γ_j are disjoint from each other and all contained in $\bigcup_{n \in N_B} \partial D(z_n, r_n)$), and of course $\Sigma = \sum_{n=1}^N 2\pi r_n < 2\pi\epsilon$. Thus the ML-estimate gives that the difference on the LHS is $\leqslant \frac{M\epsilon}{\Delta}$.

For the final limiting process, we can not shrink ϵ since we constructed γ_0 based on the fixed cover $\{D(z_n, r_n)\}_{n=1}^N$ which depended on the fixed $\epsilon > 0$, but we can shrink another "quantifier variable" $\eta > 0$. Given the above fixed $z_0 \in B \setminus S \subseteq \mathrm{int}(\gamma_0) \setminus S$, we have that for ALL $\eta > 0$ small enough there is some (finite) cover $\{D(z_n', r_n')\}_{n=1}^{N'}$ of $S \cap \mathrm{int}(\gamma_0)$ that has "length" $< \eta$ (hence lies in an η-neighborhood of S, which itself lies in $\mathrm{int}(\gamma_0)$ — we did this same argument several paragraphs ago), and has also a distance of $\geqslant \Delta > 0$ from z_0 . Forming connected components of the $D(z_n', r_n')$ and performing the "outer edge" construction and apply Ex. 2 exactly as in the previous paragraph, we get that the LHS has the bound $\left| f(z_0) - \frac{1}{2\pi i} \oint_{\gamma_0} \frac{f(z)}{z-z_0}\, dz \right| \leqslant \frac{M\eta}{\Delta}$; and this time we can take $\eta \to 0$ to get $f(z_0) = \frac{1}{2\pi i} \oint_{\gamma_0} \frac{f(z)}{z-z_0}\, dz$ for $z_0 \in B \setminus S$ as desired.

†as I mentioned in the dagger footnote on the previous page, disjointness might require a small perturbation, namely increasing the radii a tiny bit — keeping "length" $< \epsilon$ — to make two almost-connected components, i.e. when the closure of two disks $D(z_n, r_n)$ intersect at one point/"the disks are tangent", into one connected component

EDIT 11/20/21: we don't actually need the "outer edge" construction for the ball B, since we can in fact pick the radius of B s.t. ∂B does not hit any points of S. The neat argument is as follows: consider the set of "allowable" radii of B. It would be $(0, R)$ for some $R > 0$, if not for the presence of S. So small disks $D(z_n, r_n)$ around $z_n \in S$ "block out" parts of $(0, R)$. How can we be sure they don't block out all of $(0, R)$? Well, $\sum r_n < \epsilon$ so the disks $D(z_n, r_n)$ can only block out a small portion of $(0, R)$ with length/measure $< 2\epsilon$! (2ϵ since $2r_n$ is diameter). Of course we still need "outer edge" construction for other parts of the proof so this didn't make the proof any shorter/easier, but still nice.

Exercise 4

(Variant of the mean-value theorem) Let $f : U \to \mathbb{C}$ be a holomorphic function on some open subset $U \subseteq \mathbb{C}$ and let γ be a simple closed curve in U, whose interior also lies in U. We want to show that for any $z \in \mathrm{int}(\gamma)$, there exists some $w \in \mathrm{im}(\gamma)$ s.t. $\left|\frac{f(z)-f(w)}{z-w}\right| \geqslant |f'(z)|$. Let us fix such $z_0 \in \mathrm{int}(\gamma)$, and define the function

$$g(w) := \begin{cases} \frac{f(z_0)-f(w)}{z_0-w} & \text{on } U \setminus \{z_0\} \\ f'(z_0) & \text{for } w = z_0. \end{cases}$$

By the definition of holomorphicity (i.e. complex differentiability), we do have that $\lim_{w \to z_0} g(w) = f'(z_0) =: g(z_0)$, so we know that g is continuous on U and holomorphic on $U \setminus \{z_0\}$. In particular continuity at z_0 means that g is bounded near z_0, so by the Riemann removable singularity theorem (Ex. 35 from Notes 3 which we did last week) we actually have that g is holomorphic on all of U. Observe that $K := \mathrm{im}(\gamma) \cup \mathrm{int}(\gamma)$ is the complement of the open set $\mathrm{ext}(\gamma)$, and it is bounded, it is a compact set with boundary $\partial K \subseteq \mathrm{im}(\gamma)$ (because any ball around a boundary point $p \in \partial K$ must intersect non-trivially with $\mathrm{ext}(\gamma)$ *and* K, meaning p can not possibly be in $\mathrm{ext}(\gamma)$ or $\mathrm{int}(\gamma)$ since in either of those two cases any small enough ball would lie completely in $\mathrm{ext}(\gamma)$ or $\mathrm{int}(\gamma) \subseteq K$).

By the maximum (modulus) principle for holomorphic functions (see Ex. 26 of Notes 1, or Ex. 17 of Notes 3 which we did in Homework 4), we have that $|f'(z_0)| =: |g(z_0)| \leqslant \sup_{w \in K} |g(w)| = \sup_{w \in \partial K} |g(w)| \leqslant \sup_{w \in \mathrm{im}(\gamma)} |g(w)|$, and as $\mathrm{im}(\gamma)$ is compact the supremum is attained at some $w_0 \in \mathrm{im}(\gamma)$, meaning $|f'(z_0)| \leqslant |g(w_0)| = \left|\frac{f(z_0)-f(w_0)}{z_0-w_0}\right|$, as desired.

Exercise 5

(Formulas regarding Laurent expansion coefficents) Let $f : U \to \mathbb{C}$ be some holomorphic function on an open set $U \subseteq \mathbb{C}$ that contains some some annulus $A(z_0, r, R) := \{z \in \mathbb{C} : r < |z - z_0| < R\}$ where $0 \leqslant r < R \leqslant \infty$. Going off of Terry's derivation (using Lemma 1 of Notes 4) of the Laurent series in the paragraph above Ex. 5 in Notes 4, we are asked to prove ① that the Laurent series coefficients a_n are uniquely determined by f, r, R; ② that they are given by $a_n = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\, dz$ for any $n \in \mathbb{Z}$ and any simple closed curve with $W_\gamma(z_0) = 1$ lying in the annulus $A(z_0, r, R)$; and ③ the radius of convergence bounds $\liminf_{n\to\infty} |a_n|^{-1/n} \geqslant R$ and $\liminf_{n\to\infty} |a_{-n}|^{-1/n} \geqslant 1/r$.

Let us first review Terry's derivation of the Laurent series: applying Lemma 1 of Notes 4 (with say $\gamma_1 := \gamma_{z_0,R_-,\circlearrowleft}$ and $\gamma_2 := \gamma_{z_0,r_+,\circlearrowleft}$ where "R-" and "r_+" denote numbers every so slightly smaller than R and greater than r respectively, to ensure $\operatorname{im}(\gamma_1), \operatorname{im}(\gamma_2) \subseteq A(z_0, r, R)$ — it really doesn't matter since we never use γ_1, γ_2 later) we get a unique decomposition $f = f_1 + f_2$ where $f_1 : U \cup \operatorname{int}(\gamma_1) \to \mathbb{C}$ and $f_2 : U \cup \operatorname{ext}(\gamma_2) \to \mathbb{C}$ are holomorphic on their domains of definition and $f_2(z) \to 0$ as $z \to \infty$; note in particular that $D(z_0, R) \subseteq A(z_0, r, R) \cup \operatorname{int}(\gamma_1) \subseteq U \cup \operatorname{int}(\gamma_1)$ and $\overline{D(z_0, r)}^{\complement} \subseteq A(z_0, r, R) \cup \operatorname{ext}(\gamma_2) \subseteq U \cup \operatorname{ext}(\gamma_2)$. Terry then applies Cor. 20 of Notes 3 to get Taylor expansions $f_1(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ (where the RHS is absolutely convergent on $D(z_0, R) \implies$ the radius of convergence — which equals $\liminf_{n\to\infty} |a_n|^{-1/n}$ by Cauchy-Hadamard — is $\geqslant R$) and $f_2(z_0 + \frac{1}{z}) = \sum_{n=1}^{\infty} a_{-n} z^n$ (where similarly the RHS absolutely convergent on $D(0, \frac{1}{r}) \implies \liminf_{n\to\infty} |a_{-n}|^{-1/n} \geqslant 1/r$).

So we have adressed ③. For uniqueness, suppose that we have another Laurent expression $f(z) = \sum_{n=-\infty}^{\infty} b_n (z - z_0)^n$ for $z \in A(z_0, r, R)$. Then we can decompose f into the sum of $g_1(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$ and $g_2(z) = \sum_{n=-\infty}^{-1} b_n (z - z_0)^n$, which are holomorphic respectively on $D(z_0, r)$ and $\overline{D(z_0, r)}^{\complement}$ and has $g_2(z) \to 0$ as $z \to \infty$. In the previous paragraph, had we applied Lemma 1 to "$A(z_0, r, R)$" in place of "U", we would have gotten a unique decomposition $f = h_1 + h_2$ where $h_1 : A(z_0, r, R) \cup \operatorname{int}(\gamma_1) = D(z_0, R) \to \mathbb{C}$ and $h_2 : A(z_0, r, R) \cup \operatorname{ext}(\gamma_2) = \overline{D(z_0, r)}^{\complement} \to \mathbb{C}$, holomorphic on their domains of definition with $h_2(z) \to 0$ as $z \to \infty$. But both decompositions (g_1, g_2) and (f_1, f_2) satisfy these conditions, so by uniqueness of the decomposition $f = h_1 + h_2$, we have that f_1, g_1, h_1 agree everywhere on $D(z_0, R)$ and f_2, g_2, h_2 agree everywhere on $\overline{D(z_0, r)}^{\complement}$. By uniqueness of power series, i.e. Ex. 17 of Notes 1 (which we did in Homework 2) applied to f_1, g_1 and f_2, g_2 (in the latter case we might have to do the same change of variables from the previous paragraph $f_2(z_0 + \frac{1}{z}) = \sum_{n=1}^{\infty} a_{-n} z^n$ and similar for g_2) we get that $a_n = b_n$ for all $n \in \mathbb{Z}$.

We're now done with ① and ③, so to finish we prove ②. This boils down to justifying an integral-sum interchange, since if we knew we could do such an interchange, we would get (for all fixed $n \in \mathbb{Z}$) $\oint_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\, dz = \sum_{-1 \neq j \in \mathbb{Z}} \oint_\gamma a_{j+(n+1)} (z - z_0)^j\, dz + a_n \oint_\gamma \frac{1}{z - z_0}\, dz$, where the first term is 0 by the FTC(Id) (Thm. 28 in Notes 2) since all $(z - z_0)^j$ for $j \neq -1$ have antiderivatives on $\mathbb{C} \setminus \{z_0\}$, and the second term is $a_n \cdot 2\pi i W_\gamma(z_0)$ by the definition of the winding number (Def. 40 in Notes 3). To justify this integral-sum interchange, we use the Weierstrass M-test (in the manner Terry used it in Cor. 20 of Notes 3). We use the "r'-trick" we saw previously in e.g. Prop. 7 of Notes 1.

Using that disjoint compact/closed sets are distant), $\operatorname{im}(\gamma) \subseteq A(z_0, r, R)$ has positive distance from $A(z_0, r, R)^{\complement}$, so we have that for all $z \in \operatorname{im}(\gamma)$, $r < r_+ \leqslant |z - z_0| \leqslant R\text{-} < R$. We can then bound (for arbitrary fixed $n \in \mathbb{Z}$)

$$\left|\frac{1}{(z-z_0)^{n+1}}\right| \cdot \sum_{j=-\infty}^{\infty} |a_j (z - z_0)^j| \leqslant \max\{\tfrac{1}{r^{n+1}}, R^{n+1}\} \cdot \left(\sum_{j=0}^{\infty} |a_j (R\text{-})^j| + \sum_{j=-\infty}^{-1} |a_j (r_+)^j| \right),$$

where the first term in the parenthesis is finite because we know that $f_1(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ is absolutely convergent on $D(z_0, R)$, and similarly the second term in the parenthesis is finite because we know that $f_2(z) = \sum_{n=-\infty}^{-1} a_n (z - z_0)^n$ is absolutely convergent on $\overline{D(z_0, r)}^{\complement}$. Thus the Weierstrass M-test gives uniform convergence, which suffices to justify the integral-sum interchange.

Exercise 6

(Fourier inversion formula) Let $0 < r < 1 < R \leqslant \infty$. We also number the following formulas: ⑤ $\liminf_{n\to\infty} |a_n|^{-1/n} \geqslant R$; ⑥ $\liminf_{n\to\infty} |a_{-n}|^{-1/n} \geqslant 1/r$; ⑦ $f(e^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$; and ⑧ $a_n = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}\,d\theta$ (the reason for this "ad-hoc numbering" is because we are using Terry's equation numbering in Ex. 6 of Notes 4)

(i) We want to show that if f is holomorphic on $A(0, r, R)$ (notation from Ex. 5 above), then f has an absolutely convergent Fourier expansion ⑦ with coefficients that obey the formula ⑧ and the bound ⑤ and ⑥.

This is literally just Ex. 5 above (which gives that f has an absolutely convergent Laurent series $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ on $A(r, R)$), but plugging in $z = e^{i\theta}$ and using Ex. 17(iv) from Notes 2 (Homework 3) on $\gamma = \gamma_{0,1,\circlearrowleft}$. More explicitly, $\gamma = \gamma_{0,1,\circlearrowleft}$ lies in $A(0, r, R)$ and has winding number 1, so the Laurent series coefficient formula (② from Ex. 5 above) gives

$$a_n = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{(z-0)^{n+1}}\,dz = \frac{1}{2\pi i}\oint_0^{2\pi} \frac{f(e^{i\theta})}{(e^{i\theta})^{n+1}}(ie^{i\theta})\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}\,d\theta,$$

as desired.

(ii) Conversely, suppose we are given $a_n \in \mathbb{C}$ (for all $n \in \mathbb{Z}$) that satisfy the asymptotic Cauchy-Hadamard formula bound ⑤ and ⑥. We want to show that there is a function f (namely $f(z) := \sum_{n=-\infty}^{\infty} a_n z^n$) holomorphic on $A(0, r, R)$ having the Fourier expansion ⑦ with coefficients obeying ⑧ ("the inversion formula"). It suffices to prove that $f(z) := \sum_{n=-\infty}^{\infty} a_n z^n$ is holomorphic on $A(r, R)$, since we already proven in part (i) that such functions obey ⑦ and ⑧ (and since ⑦ and ⑧ are based off the Laurent expansion, it is key that Laurent expansions are unique, but of course we proved this already in ① of Ex. 5 above).

Well, $f_1(z) := \sum_{n=0}^{\infty} a_n z^n$ is a power series with radius of convergence (by Cauchy-Hadamard, Prop. 7 in Notes 1) $\liminf_{n\to\infty} |a_n|^{-1/n} \geqslant R$ and hence holomorphic (by Thm. 15 of Notes 1 — term by term differentiation) on $D(0, R)$; and similarly $f_2(z) := \sum_{n=1}^{\infty} a_{-n} z^n$ is a power series of radius of convergence and $\liminf_{n\to\infty} |a_{-n}|^{-1/n} \geqslant 1/r$ hence holomorphic on $D(0, 1/r)$, implying that $f_2(\frac{1}{z}) = \sum_{n=-\infty}^{-1} a_n z^n$ is holomorphic on $\overline{D(z_0, r)}^{\complement}$. Therefore, their sum $f(z) = f_1(z) + f_2(\frac{1}{z})$ is holomorphic on $A(0, r, R)$, as desired.

Exercise S&S §3.9

We are asked to evaluate $\int_0^1 \log(\sin \pi x)\,dx$. Amazingly, it is easier to do this using real-analytic methods than complex-analytic methods. First we show that $\int_0^{1/2} \log(\sin \pi x)\,dx$ exists. By convexity of $\sin(\pi x)$ on $[0, 1]$ (as can be verified by computing the derivative — I already gave a spiel about this in Exercise S&S §2.1 of Homework 4), $\sin \pi x$ lies above its secant line $L(x) = 2x$ on $[0, \frac{1}{2}]$ ("secant line on $[0, \frac{1}{2}]$" means that $L(x)$ and $\sin \pi x$ share the same values at $x = 0, \frac{1}{2}$). Note that both functions are $\leqslant 1$ on $[0, \frac{1}{2}]$.

Then since $\log x$ is increasing on $(0,\infty)$ and $\log(1)=0$, we have that $0 \geqslant \log(\sin \pi x) \geqslant \log(2x)$, and so $\int_0^{1/2} |\log(\sin \pi x)|\,dx = \lim_{\epsilon\to 0}\int_\epsilon^{1/2} -\log(\sin \pi x)\,dx \leqslant -\lim_{\epsilon\to 0}\int_\epsilon^{1/2}\log(2x)\,dx = -\lim_{\epsilon\to 0}(x\log 2x - 2x)|_\epsilon^1 = 2-\log(2)$ (using that $\lim_{x\to 0} x\ln x = 0$ by say l'Hôpital's rule); and of course monotone decreasing sequences with lower bounds like $\lim_{\epsilon\to 0}\int_\epsilon^{1/2}\log(\sin \pi x)\,dx \geqslant \log 2 - 2$ must converge (monotone decreasing since $\log(\sin \pi x) \leqslant 0$ on $(0, \frac{1}{2}]$), so indeed $\int_0^{1/2}\log(\sin \pi x)\,dx = \lim_{\epsilon\to 0}\int_\epsilon^{1/2}\log(\sin \pi x)\,dx$ exists and is finite, say I.

For the next part, recall the trig identities $\sin x = \sin(\pi - x)$, $\sin x = \cos(\frac{\pi}{2} - x)$, and $\sin(2x) = 2\sin x \cos x$ for all $x \in \mathbb{R}$. We may now make several variable changes (in the next bit, let us assume we are using the Lebesgue-Stieltjes integral to avoid improper Riemann integrals). Let $u = 1-x \implies$ "$du = -\,dx$"; then $\int_0^{1/2}\log(\sin \pi x)\,dx = \int_1^{1/2}\log(\sin(\pi - \pi u))(-\,du) = \int_{1/2}^1 \log(\sin \pi u)\,du$. Thus our desired integral is $\int_0^1 \log(\sin \pi x)\,dx = 2I$. We can also let $v = \frac{1}{2} - x \implies$ "$dv = -\,dx$"; then $I = \int_0^{1/2}\log(\sin \pi x)\,dx = \int_{1/2}^0 \log(\sin(\frac{\pi}{2} - \pi v))(-\,dv) = \int_0^{1/2}\log(\cos \pi v)\,dv$.

Summing we get $2I = \int_0^{1/2}\log(\sin \pi x)\,dx + \int_0^{1/2}\log(\cos \pi x)\,dx = \int_0^{1/2}\log(\sin(\pi x)\cos(\pi x))\,dx = \int_0^{1/2}\log(\frac{1}{2}\sin(2\pi x))\,dx$. Making the final variable change $w = 2x \implies$ "$dw = 2\,dx$", we get that this integral is equal to $\int_0^1 \log(\frac{1}{2}) + \log(\sin \pi w)(\frac{1}{2}\,dw) = \log(\frac{1}{2}) + \frac{1}{2}\cdot 2I = I - \log 2$. In other words, we have $2I = I - \log 2 \implies I = -\log 2$, and we are done.

Exercise S&S §3.12

Suppose u is not an integer. We are asked to prove that $\sum_{n=-\infty}^{\infty}\frac{1}{(u+n)^2} = \frac{\pi^2}{(\sin \pi u)^2}$ by integrating $f(z) := \frac{\pi \cot \pi z}{(u+z)^2}$ over the circle $[|z| = R_N]$ for $R_N := N + \frac{1}{2}$ ($N \in \mathbb{N}$ larger than $|u|$), adding the residues of f inside the circle, and letting $N \to \infty$. We recall the residue theorem for a simple counterclockwise closed curve (Cor. 23 in Notes 4): if γ is a simple counterclockwise closed curve avoiding the closed discrete set S of singularities, with image and interior lying in an open set U on which $f : U \setminus S \to \mathbb{C}$ is holomorphic, then $\oint_\gamma f(z)\,dz = 2\pi i \sum_{z_0 \in S\cap \mathrm{int}(\gamma)} \mathrm{Res}[f, z_0]$.

Also, Ex. 26 of Notes 4 (which we'll do next week, not depending on this problem so no qualms about circularity) says that if f has a simple pole at z_0, then $\mathrm{Res}[f, z_0] = \lim_{z\to z_0} f(z)(z - z_0)$ where this limit exists and is a complex number, i.e. not ∞ or something (the limit is taken with $z \notin U \setminus S$ and $z \neq z_0$); and more generally if $f(z) = \frac{g(z)}{(z-z_0)^m}$ near z_0 for some g holomorphic at z_0, then $\mathrm{Res}[f, z_0] = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}} g(z_0)$.

With these two paragraphs of stage-setting, we add one more piece of information: if f has a zero of order $m \geqslant 1$ at z_0 and g is holomorphic and non-zero at z_0, then $\frac{g(z)}{f(z)}$ has a pole of order m at z_0 — one can see this using the procedure for dividing infinite series using the geometric series formula (we can first multiply by a constant $\lambda \in \mathbb{C}$ to make $a_m = 1$, and translate so that $z_0 = 0$):

$$\frac{1}{f(z)} = \lambda \cdot \frac{1}{\lambda f(z)} = \frac{\lambda}{z^m} \cdot \frac{1}{1 - (\sum_{n=m+1}^{\infty} a_n z^{n-m})}$$
$$= \frac{\lambda}{z^m} \cdot \left[1 + \left(\sum_{n=m+1}^{\infty} a_n z^{n-m} \right)^1 + \left(\sum_{n=m+1}^{\infty} a_n z^{n-m} \right)^2 + \dots \right]$$

and then using the procedure to multiply infinite series to multiply with $g(z) = \sum_{n=0}^{\infty} b_n z^n$ (more details found on pg. 153 of Gamelin). We are ready to apply the above results to our particular situation. First recall that the zeroes of $\sin(z)$ in $\mathbb{C}$ are precisely at the points $\pi\mathbb{Z}$, and one can see easily using the Taylor series that these zeroes are simple. Similarly, the zeroes (also all simple) of $\cos(z)$ in $\mathbb{C}$ are precisely the points $\pi\mathbb{Z} + \frac{\pi}{2}$.

Thus, the poles of $f = \pi \cdot \frac{1}{(u+z)^2} \cdot \frac{\cos(\pi z)}{\sin(\pi z)}$ in $\mathbb{C}$ are: poles of order exactly 1 at $\mathbb{Z}$ ("exactly 1" because $-u$ is not an integer); and a pole of order $\leqslant 2$ at $-u$ ("$\leqslant 2$" since u could be zero of cosine, i.e. in $\mathbb{Z} + \frac{\pi}{2}$ in which case we'd have a pole of order 1). Let us focus on $z_0 = k \in \mathbb{Z}$ first. Note that $\frac{z-z_0}{\sin(\pi z)} = \frac{1}{\pi} \frac{\pi z - \pi z_0}{\sin(\pi z) - \sin(\pi z_0)}$ is the reciprocal of $\pi \cdot \frac{\sin(\pi z) - \sin(\pi z_0)}{\pi z - \pi z_0}$, and this latter quantity has a limit as $z \to z_0$, namely it is π times the derivative of $\sin(z)$ at πz_0, i.e. $\cos(\pi z_0)$; by limit arithmetic rules we have that $\lim_{z \to z_0} \frac{z-z_0}{\sin(\pi z)} = \frac{1}{\pi \cos(\pi z_0)}$. Therefore,

$$\text{Res}[f, z_0] = \lim_{z \to z_0} f(z)(z - z_0) = \pi \cdot \frac{1}{(u + z_0)^2} \cdot \cos(\pi z_0) \cdot \lim_{z \to z_0} \frac{z - z_0}{\sin(\pi z)} = \frac{1}{(u + z_0)^2}.$$

The remaining singularity is $z_0 = -u$, and we know (from the 2nd paragraph I wrote for this exercise) that $\text{Res}[f, z_0] = \frac{d}{dz} g(z_0)$ where $g(z) = \pi \cot(\pi z)$ is indeed holomorphic at $-u$ (again it is key that $-u \notin \mathbb{Z}$). Calculating this derivative we have that

$$\text{Res}[f, z_0] = \frac{-\pi^2}{\sin(\pi z_0)^2}.$$

Then, since we chose R_N to be in $\mathbb{Z} + \frac{1}{2}$ and larger than $|u|$, the curve $\gamma_N := \gamma_{0,R_N,\circlearrowleft}$ avoids all singularities, so by the residue theorem, $\oint_{\gamma_N} f(z)\, dz = 2\pi i \sum_{z_0 \in S \cap B(0,R_N)} \text{Res}[f, z_0]$. The last step is then to see that $\oint_{\gamma_N} f(z)\, dz$ goes to 0 as $N \to \infty$. The ML-estimate gives that

$$\oint_{\gamma_N} f(z)\, dz \leqslant 2\pi R_N \cdot \frac{1}{|u + R_N|^2} \cdot \max_{z \in \text{im}(\gamma_N)} \left| \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \right|$$

where $\left| \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \right|$ can be found to be bounded by some constant on $\text{im}(\gamma_N)$ for all large natural numbers N. Taking the limit as $N \to \infty$, we get

$$0 = \sum_{k \in \mathbb{Z}} \frac{1}{(u + k)^2} - \frac{\pi^2}{(\sin \pi u)^2},$$

which gives us the desired identity.

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Daniel Rui - 11/5/21

Exercise 35 (Notes 3)

(Riemann removable singularity theorem) Let $U \subseteq \mathbb{C}$ be open, $z_0 \in U$, and $f : U \setminus \{z_0\} \to \mathbb{C}$ be holomorphic that is furthermore bounded near z_0, in the sense that for some $r > 0$ there is some bound B s.t. $|f(z)| \leqslant B$ on $D(z_0, r) \setminus \{z_0\}$. We are asked to prove that then f can be extended to a holomorphic function $\tilde{f} : U \to \mathbb{C}$; in other words we want to show that we can "remove the singularity" at z_0.

Let us define $g(z) := \begin{cases} (z-z_0)f(z) & \text{on } U \setminus \{z_0\} \\ 0 & \text{for } z = z_0 \end{cases}$, or in other words let g be the continuous extension of $(z - z_0)f(z)$ to U (continuous and in fact holomorphic on $U \setminus \{z_0\}$ as it is a product of holomorphic functions, and continuous at z_0 because for every $\epsilon > 0$, choosing $\delta = \frac{1}{B}\min\{\epsilon, r\}$ yields $|z - z_0| < \delta \implies |g(z) - 0| \leqslant |z - z_0|B < \epsilon$). Note that on $U \setminus \{z_0\}$, $f(z) = \frac{g(z)-g(z_0)}{z-z_0}$, and so f is continuous at $z_0 \iff g$ is differentiable at z_0, in which case extending $\tilde{f}(z_0) := g'(z_0)$ (of course $\tilde{f} :\equiv f$ on $U \setminus \{z_0\}$) would make $\tilde{f} : U \to \mathbb{C}$ a continuous function. It is now clear that we really want to show that these continuous functions (first g, then $\tilde{f}$) on U are actually holomorphic, so we turn to Morera's theorem.

Morera's theorem (as it appears in Theorem 33 in Notes 3) says that for any open $U \subseteq \mathbb{C}$ and continuous $f : U \to \mathbb{C}$, if f is conservative for all polygonal paths, i.e. $\oint_\gamma f(z)\,dz = 0$ for all closed polygonal paths $\gamma \subseteq U$, then f is holomorphic on U. In our proof of Exercise 32 from Notes 2 (which we did last week), we in particular proved that f being conservative for all simple closed triangles (a further weakening of condition (iii) from Ex. 32) $\implies$ f is conservative for all polygonal paths (condition (ii) from Ex. 32), so Morera's theorem holds even if we just assume f is conservative for all simple closed triangles.

Thus to prove g is holomorphic at z_0, it is more than enough to show that g is conservative for all triangular paths $\gamma_\triangle$ in some small disk $D(z_0, r) \subseteq \overline{D(z_0, r)} \subseteq U$. If $\mathrm{im}(\gamma_\triangle) \cup \mathrm{int}(\gamma_\triangle) = \overline{\mathrm{hull}(\gamma_\triangle)}$ lies in $D(z_0, r) \setminus \{z_0\}$ ("hull" here denotes the open convex hull of the 3 vertices of the triangular path $\gamma_\triangle$; I use this notation to emphasize that for triangles, unlike general polygons/curves, the interior is very easy to work with as it is precisely the convex combinations/weighted averages of the vertices), then by convexity of $\overline{\mathrm{hull}(\gamma_\triangle)}$ we can contract it to any point inside the (hull of the) triangle, so by Cauchy's theorem applied to $\gamma_\triangle \subseteq D(z_0, r) \setminus \{z_0\}$ (an open set on which g is holomorphic), $\oint_{\gamma_\triangle} g(z)\,dz = 0$.

If $z_0 \in \mathrm{hull}(\gamma_\triangle) = \mathrm{int}(\gamma_\triangle)$ (an open set), then for all $\epsilon > 0$ small enough $\gamma_{z_0,\epsilon,\circlearrowleft}$ lies in $\mathrm{int}(\gamma_\triangle)$, and all such curves $\gamma_\triangle, \gamma_{z_0,\epsilon,\circlearrowleft}$ live in $D(z_0, r) \setminus \{z_0\}$ and are homotopic as closed curves (perhaps up to reparameterization, or some "walking" of the endpoint of the closed curve along the closed curve) by the straight-line homotopy/radial projection (which in particular stays in $\overline{\mathrm{hull}(\gamma_\triangle)} \setminus D(z_0, \epsilon) \subseteq D(z_0, r) \setminus \{z_0\}$). Thus again by Cauchy's theorem $\oint_{\gamma_\triangle} g(z)\,dz = \oint_{\gamma_{z_0,\epsilon,\circlearrowleft}} g(z)\,dz$ for all small enough $\epsilon > 0$. By the ML-estimate, since g is continuous on U we have that on the compact $\overline{D(z_0, r)} \subseteq U$ it is bounded by say M, so $|\oint_{\gamma_{z_0,\epsilon,\circlearrowleft}} g(z)\,dz| \leqslant M \cdot 2\pi\epsilon \to 0$ as $\epsilon \to 0$. Thus $\oint_{\gamma_\triangle} g(z)\,dz = 0$ in this case as well.

Finally if $z_0 \in \text{im}(\gamma_\triangle)$ (the last case, since we did the case $z_0 \in \text{ext}(\gamma_\triangle)$ and $z_0 \in \text{int}(\gamma_\triangle)$ above already), we just perturb the triangle slightly $\gamma_{\triangle+h}(t) := \gamma_\triangle(t) + h$ for some small complex number $h \in \mathbb{C}$ so that $\gamma_{\triangle+h}$ is covered by one of the above cases, and use the uniform convergence of such paths as $h \to 0$ (described/discussed in much greater detail in the next exercise, Ex. 37 below) to get that $\oint_{\gamma_\triangle} g(z)\,dz$ also equals 0 in this case.

We have now successfully used Morera's theorem to show that g is holomorphic at z_0, which recall is equivalent to $\tilde{f}$ being continuous at z_0 (where we defined $\tilde{f}$ to be an extension of f with $\tilde{f}(z_0) := g'(z_0)$). Finally, applying the paragraphs above to $\tilde{f}$ (both $\tilde{f}, g$ are continuous on U and holomorphic on $U \setminus \{z_0\}$, and that's all we used!), Morera's theorem proves that $\tilde{f}$ is holomorphic at z_0, and so we are done.

Remark: notice that with this proof, we may weaken the assumption that f be bounded; we merely need $(z - z_0)f(z)$ to approach 0 as $z \to z_0$. In fact as we learned in class 11/5/21, for non-essential isolated singularities f can only approach ∞ at a "rate of $(z - z_0)^{-n}$ for $n \in \mathbb{N}$, e.g. so here if we know $|f|$ approaches ∞ "slower than at a rate of $|z - z_0|^{-1/2}$" as $z \to z_0$, then it in fact must be bounded!

Exercise 37

(Schwarz reflection principle) Let $U \subseteq \mathbb{C}$ be open and symmetric across $\mathbb{R}$ (i.e. $z \in U \iff \overline{z} \in U$), and let $U^+ := \{z \in U : \mathfrak{Im}(z) > 0\}$ and similarly $U^- := \{z \in U : \mathfrak{Im}(z) < 0\}$. Suppose that f^+, f^- are two functions s.t. $f^\pm : \overline{U^\pm} \to \mathbb{C}$ are continuous and holomorphic on $U^\pm$ respectively, which furthermore agree on $U \cap \mathbb{R}$. We want to show that gluing together $f^\pm$ yields a holomorphic function $f : U \to \mathbb{C}$.

f is holomorphic on $U \setminus \mathbb{R}$, so we just need to show that f is holomorphic at every $z_0 \in U \cap \mathbb{R}$. Morera's theorem (as I wrote it in Ex. 35 bove) says that for any open $U \subseteq \mathbb{C}$ and continuous $f : U \to \mathbb{C}$, if f is conservative for all triangular paths, i.e. $\oint_\gamma f(z)\,dz = 0$ for all simple closed triangular paths $\gamma \subseteq U$, then f is holomorphic on U. Thus to prove f is holomorphic at z_0, it is more than enough to show that f is conservative for all triangular paths $\gamma_\triangle$ in $D(z_0, r) \subseteq U$. Let a, b, c be the vertices of $\gamma_\triangle$ (i.e $\gamma_\triangle = [a \to b \to c \to a]$). Let $D(z_0, r)^+ = D(z_0, r) \cap U^+$ and similarly for $D(z_0, r)^-$. Both are convex open sets, so in particular $\oint f$ evaluated over any polygonal path in either is going to yield 0. Thus, the only case left is if $\gamma_\triangle$ intersects $\mathbb{R}$.

There must be at least one vertex in each open "side" $D(z_0, r)^+, D(z_0, r)^-$ (since we are assuming the triangle is not degenerate, i.e. the points can't all lie on the line $\mathbb{R}$) and because we are assuming $\gamma_\triangle$ intersects $\mathbb{R}$, there must be at least one vertex in the opposite "side" $\overline{D(z_0, r)^-}, \overline{D(z_0, r)^+}$ (respectively). By the pigeonhole principle it must be split 2 on one "side" and 1 on the other, so say w.l.o.g. that a, c are on the same "side" (and b is on the other "side") and $\mathbb{R}$ intersects $\gamma_\triangle$ at the line segments $[a \to b]$ and $[b \to c]$ at intersection points m_{ab}, m_{bc} ("w.l.o.g." because we saw in our proof of Ex. 32 of Notes 2 in Homework 3 that we can permute vertices of a triangular path without changing the integral). Then either $\{a, c, m_{ab}, m_{bc}\} \subseteq \overline{D(z_0, r)^+}$ and $\{b, m_{ab}, m_{bc}\} \subseteq \overline{D(z_0, r)^-}$ or vice versa.

Then define the "quadrilateral path" $\gamma_Q := [a \to m_{ab} \to m_{bc} \to c \to a]$ and the "smaller triangular path" $\gamma_T := [m_{ab} \to b \to m_{bc} \to m_{ab}]$. Then $\oint_{\gamma_\triangle} = \int_{[a\to m_{ab}]+[m_{ab}\to b]} + \int_{[b\to m_{bc}]+[m_{bc}\to c]} \pm \int_{[m_{ab}\to m_{bc}]} + \int_{[c\to a]} = \oint_{\gamma_T} + \oint_{\gamma_Q}$ (we did a very similar integral splitting/recombining in our proof of Ex. 32 of Notes 2 in Homework 3). Suppose now $T \subseteq \overline{D(z_0,r)^+}$ (otherwise just swap $+i\epsilon$ with $-i\epsilon$ in what follows), and define $\gamma_{T+i\epsilon}(t) := \gamma_T(t) + i\epsilon$ and similarly $\gamma_{Q-i\epsilon}$. These paths now lie in $D(z_0,r)^+$ and $D(z_0,r)^-$ respectively, and converge uniformly to γ_T, γ_Q as $\epsilon \to 0$.

To rigorously prove that $\oint_{\gamma_{T+i\epsilon}} + \oint_{\gamma_{Q-i\epsilon}}$ converges to $\oint_{\gamma_T} + \oint_{\gamma_Q} = \oint_{\gamma_\triangle}$, we need a result about uniform convergence of paths yielding convergence of integrals. We can proceed using the original definition of Riemann sums, observing that

$$\begin{aligned}\varepsilon(t_{j-1},t_j) &:= |f(\gamma_n(t_j))(\gamma_n(t_j) - \gamma_n(t_{j-1})) - f(\gamma(t_j))(\gamma(t_j) - \gamma(t_{j-1}))| \\ &= |f(\gamma_n(t_j)(\gamma_n(t_j) - \gamma_n(t_{j-1})) \pm f(\gamma(t_j))(\gamma_n(t_j) - \gamma_n(t_{j-1})) - f(\gamma(t_j))(\gamma(t_j) - \gamma(t_{j-1}))| \\ &\leqslant |f(\gamma_n(t_j)) - f(\gamma(t_j))||\gamma_n(t_j) - \gamma_n(t_{j-1})| + |f(\gamma(t_j))|\Big(|\gamma_n(t_j) - \gamma(t_j)| + |\gamma_n(t_{j-1}) - \gamma(t_{j-1})|\Big)\end{aligned}$$

and so summing yields

$$\left|\sum_{j=1}^{m} f(\gamma(t_j))(\gamma(t_j) - \gamma(t_{j-1})) - \sum_{j=1}^{m} f(\gamma_n(t_j))(\gamma_n(t_j) - \gamma_n(t_{j-1}))\right| \leqslant \sum_{j=1}^{m} \varepsilon(t_{j-1},t_j) \leqslant \epsilon_n|\gamma_n| + M \cdot 2\epsilon_n$$

where ϵ_n comes from uniform convergence of $\gamma_n \rightrightarrows \gamma$, i.e. $\epsilon_n > 0$ satisfies $n \geqslant N \implies \|\gamma_n - \gamma\|_\infty < \epsilon_N$, and M is the bound of $|f|$ on the compact set $\operatorname{im}(\gamma)$. The above difference goes $\to 0$ as $n \to \infty$, assuming that all the $|\gamma_n|$ are uniformly bounded (which they are in our case because we're just translating so $|\gamma_n| = |\gamma|$). Thus the integrals $\int_{\gamma_n} f(z)\,dz \to \int_\gamma f(z)\,dz$. **Alternatively**, in our case since we are just translating the paths γ_T and γ_Q, instead of "moving the paths", we can "move f" by the translations $f(z \pm i\epsilon)$, and of course uniform convergence of functions $\rightsquigarrow$ convergence of integrals.

Bringing everything together, we had that $\gamma_{T+i\epsilon}$ and $\gamma_{Q-i\epsilon}$ lie in $D(z_0,r)^+$ and $D(z_0,r)^-$ respectively (for $\epsilon > 0$ small enough, since $D(z_0,r)^+$ and $D(z_0,r)^-$ are open sets; here we again using that disjoint compact/closed sets are distant). As f is holomorphic on those regions and those regions are convex (hence simply connected), $\gamma_{T+i\epsilon}$ and $\gamma_{Q-i\epsilon}$ are contractible in their respective regions and so by Cauchy's theorem $\oint_{\gamma_{T+i\epsilon}} + \oint_{\gamma_{Q-i\epsilon}} = 0 + 0$. Our uniform convergence result above then yields $\oint_{\gamma_\triangle} = \oint_{\gamma_T} + \oint_{\gamma_Q} = 0$, and so by Morera's theorem, f is holomorphic on all of U and we are done.

Part (ii) (Corollary)

Suppose now that f^+ is real-valued on $U \cap \mathbb{R}$ (still continuous on $\overline{U^+}$ and holomorphic on U^+). We are asked to show that defining $f^-(z) := \overline{f^+(\overline{z})}$ for $z \in \overline{U^-}$ (which is in particular continuous on $\overline{U^-}$ since f and conjugation are both continuous) and gluing f^+ and f^- together (continuous functions defined on closed subsets that agree on their intersections can be glued together gluing/pasting lemma for general topological spaces; in our case $U \cap \mathbb{R}$ is the set of intersection, and because f^+ is real valued on $U \cap \mathbb{R}$, we have for any real number $x \in U \cap \mathbb{R}$ that $f^-(x) = \overline{f^+(\overline{x})} = f^+(x)$).

It is clear that we would be done by the above general Schwarz reflection principle if we knew that f^- was holomorphic on U^-. This is an easy computation: abbreviating f^+ by f, we have (using that for $a, b \neq 0$, $\overline{(\frac{a}{b})} = \frac{|a/b|^2}{a/b} = \frac{|a|^2}{a}\frac{b}{|b|^2} = \frac{\overline{a}}{\overline{b}}$, where obviously the formula still holds for $a = 0$; and of course $\overline{z+w} = \overline{z} + \overline{w}$)

$$\frac{\overline{f(\overline{z})} - \overline{f(\overline{z_0})}}{z - z_0} = \overline{\left(\frac{f(\overline{z}) - f(\overline{z_0})}{\overline{z} - \overline{z}_0}\right)}$$

implying that

$$\lim_{z \to z_0} \frac{\overline{f(\overline{z})} - \overline{f(\overline{z_0})}}{z - z_0} = \overline{\lim_{\overline{z} \to \overline{z_0}} \left(\frac{f(\overline{z}) - f(\overline{z_0})}{\overline{z} - \overline{z}_0}\right)} = \overline{\lim_{w \to \overline{z_0}} \left(\frac{f(w) - f(\overline{z_0})}{w - \overline{z}_0}\right)} = \overline{f'(\overline{z_0})}.$$

This tells us that because f^+ is holomorphic on U^+, f^- is holomorphic on U^-.

Exercise 40

(Higher-order generalized Cauchy integral formula) Let $U \subseteq \mathbb{C}$ be simply connected, γ be a closed curve in U, and $f : U \to \mathbb{C}$ be holomorphic. We are asked to show that ($W_\gamma(z_0)$ refers to the winding number $W_\gamma(z_0) := \frac{1}{2\pi i}\oint_\gamma \frac{1}{z-z_0}\,dz$)

$$W_\gamma(z_0) f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz.$$

One might think to proceed by the way we proved the higher-order Cauchy integral formulas in Corollary 18 in Notes 3, namely to argue that because γ is simply connected, we can (homotopically) contract it to $\tilde{\gamma}$ which lies in $D(z_0, r)$ (assuming that $z_0 \in \operatorname{int}(\gamma)$) and then to use $\frac{1}{z-w} = \sum_{n=0}^{\infty} \frac{1}{(z-z_0)^{n+1}}(w-z_0)^n$ (for $w \in D(z_0, r)$) to get $\frac{1}{2\pi i}\oint_{\gamma'} \sum_{n=0}^{\infty} \frac{f(z)}{(z-z_0)^{n+1}}(w-z_0)^n\,dz = f(w) = \sum_{n=0}^{\infty} a_n(w-z_0)^n$ (again doing sum-integral interchange by Weierstrass M-test or DCT, just as in Cor. 18 of Notes 3). However there are some issues regarding contractability, i.e. how can we be sure that the contraction doesn't cross z_0, since we only know that we need $\gamma, \tilde{\gamma}$ to be homotopic in $U \setminus \{z_0\}$, the open set on which $\frac{f(z)}{(z-z_0)^{n+1}}(w-z_0)^n$ is holomorphic.

The actual route we take is in the manner of Thm. 39 in Notes 3 (generalized Cauchy integral formula), where we we used the factor theorem/Riemann removable singularity theorem to get that $f(z) - f(z_0) = (z - z_0)g(z)$ for some holomorphic $g : U \to \mathbb{C}$, and so $\oint_\gamma \frac{f(z)-f(z_0)}{z-z_0}\,dz = \int_\gamma g(z)\,dz = 0$ (last equality by Cauchy's theorem, since γ is a closed curve in simply connected U), implying that $\oint_\gamma \frac{f(z)}{z-z_0}\,dz = \oint_\gamma \frac{f(z_0)}{z-z_0}\,dz = 2\pi i W_\gamma(z_0) f(z_0)$. We apply this method, but using higher-order approximations (i.e. Taylor series).

We know (from the Cauchy integral formula, $\frac{1}{z-w} = \sum_{n=0}^{\infty} \frac{1}{(z-z_0)^{n+1}}(w-z_0)^n$, and sum-integral interchange $\rightsquigarrow$ Taylor series trick I described above in my summary of Cor. 18) that $f(z)$ has a Taylor series $f(z) = \sum_{m=0}^{\infty} a_m(z-z_0)^m$ where $a_m = \frac{f^{(m)}(z_0)}{m!}$ that holds inside any $D(z_0, r)$ s.t. $\overline{D(z_0, r)} \subseteq U$ (so in particular the radius of convergence R is > 0 because U is open).

Then $S_n(z) := f(z) - [\sum_{m=0}^{n} a_m(z-z_0)^m] = \sum_{m=n+1}^{\infty} a_m(z-z_0)^m = (z-z_0)^{n+1}g(z)$ where g, defined by gluing together $\sum_{m=n+1}^{\infty} a_m(z-z_0)^{m-(n+1)}$ on $D(z_0,R)$ and $\frac{S_n(z)}{(z-z_0)^{n+1}}$ on $U \setminus \{z_0\}$, is holomorphic on all of U (cf. the factor theorem, Cor. 22 in Notes 3) — this gluing is possible by the identity theorem (the two pieces are holomorphic on open sets, and they agree on the open set $D(z_0,r) \setminus \{z_0\}$). By the same logic as the 0th-order case (covered above), we have $\oint_\gamma \frac{S_n(z)}{(z-z_0)^{n+1}}\,dz = \oint_\gamma g(z)\,dz = 0 \implies \oint_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \sum_{m=1}^{n} a_m \oint_\gamma (z-z_0)^{m-(n+1)}\,dz$.

For $m \in \{1,\ldots,n-1\}$, the integrand $(z-z_0)^{m-(n+1)}$ has an anti-derivative on $U \setminus \{z_0\}$ (namely $\frac{1}{m-(n+1)+1}(z-z_0)^{m-(n+1)+1}$ where $m-(n+1)+1 \in \{-n+1,\ldots,-1\}$), so by the FTC(Id) (Thm. 28 in Notes 2) which applies because $\gamma \subseteq U \setminus \{z_0\}$,

$$\oint_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \sum_{m=1}^{n} \frac{f^{(m)}(z_0)}{m!} \oint_\gamma \frac{(z-z_0)^m}{(z-z_0)^{n+1}}\,dz = \frac{1}{n!}\oint_\gamma \frac{f^{(n)}(z_0)}{z-z_0}\,dz = \frac{2\pi i}{n!} f^{(n)}(z_0) W_\gamma(z_0),$$

and we are done.

Exercise 47

We want to give a more analytic proof of the fact that winding numbers ($W_\gamma(z_0) := \frac{1}{2\pi i}\oint_\gamma \frac{1}{z-z_0}\,dz$) are integers. The proof given in Lemma 46 of Notes 3 essentially uses the "walking-the-dog theorem" to deform γ into a closed polygonal path without changing the winding number around/at z_0, and then splitting into triangles $[z_1 \to z_i \to z_{i+1} \to z_1]$ (we don't have anything to worry about except z_0 since $\frac{1}{z-z_0}$ is holomorphic on $\mathbb{C} \setminus \{z_0\}$, and even if z_0 lies on one of added lines we can perturb z_0 a little bit since winding numbers are locally constant; see Lemma 44 in Notes 3 — essentially a corollary of the "walking-the-dog theorem"), and then on each triangle using either the Cauchy integral formula (if $z_0 \in \mathrm{int}([z_{1\to i\to i+1\to 1}])$) or Cauchy's theorem (if $z_0 \in \mathrm{ext}([z_{1\to i\to i+1\to 1}])$).

For the more analytic proof, we again deform γ to a closed polygonal path (with no change to the winding number around/at z_0) and examine $\phi(t) := \exp(\int_a^t \frac{\gamma'(s)}{\gamma(s)-z_0}\,ds)/(\gamma(t)-z_0)$ defined on $[a,b]$ (note that if γ is taken to move uniformly on each line segment of the polygon, then γ' is defined on all but finitely many points — corresponding to the vertices of the polygon — and constant in between those finitely many points). We are given the hint to prove that ϕ is constant by proving it is continuous and that its derivative is 0 at all but finitely many points.

Do note that by Ex. 17(iv) from Notes 2 (which we proved in Homework 3), splitting γ into piecewise linear parts and recombining later we get that $I(t) := \int_{\gamma|_{[a,t]}} \frac{1}{z-z_0}\,dz = \int_a^t \frac{1}{\gamma(s)-z_0}\cdot\gamma'(s)\,ds$ for every $t \in [a,b]$. Showing that $W_\gamma(z_0) \in \mathbb{Z}$ is equivalent to showing $I(b) \in 2\pi i\mathbb{Z}$, which is equivalent to $\exp(I(b)) = 1$ (this is because $|e^z| = |e^x e^{iy}| = e^x$ implies that $|e^z| = 1 \implies z \in i\mathbb{R}$, and then we use e^{it} is periodic of period 2π). Of course $\phi(t) = \frac{\exp(I(t))}{\gamma(t)-z_0}$, and if we prove that $\phi(t) = C$ on $[a,b]$, then $C = \phi(a) = \frac{\exp(0)}{\gamma(a)-z_0}$, so $\exp(I(b)) = C(\gamma(b)-z_0) = \frac{\gamma(b)-z_0}{\gamma(a)-z_0} = 1$, as required (using that $\gamma(a) = \gamma(b)$). Thus indeed it suffices to prove that $\phi(t)$ is constant on $[a,b]$, as the hint suggested.

First recall the FTC(dI) on $\mathbb{R}$: for $f : [a,b] \to \mathbb{R}$, $F(t) := \int_a^t f(x)\,dx$ satisfies $F(t+h) - F(t) = \int_t^{t+h} f(x)\,dx$ which is between $\min_{x\in[t,t+h]} f(x) \cdot h$ and $\max_{x\in[t,t+h]}$. If f is bounded on $[a,b]$, then $F(t+h) \to F(t)$ as $h \to 0$, giving us continuity; and if f is continuous at t, then $\lim_{h\to 0} \frac{F(t+h)-F(t)}{h} = f(t)$ since continuity implies that on small enough intervals around t, the difference between the max and min of f gets arbitrarily small (in particular max and min get arbitrarily close to $f(t)$).

Thus (splitting into real and imaginary parts) the FTC(dI) as described above tells us that $I(t)$ is continuous on $[a,b]$, and differentiable at all points of $[a,b]$ except a finite set $\{a = t_0, \dots, t_n = b\}$ corresponding to the vertices of the polygonal curve $\gamma(t)$, where furthermore $I'(t)$ at points of differentiability is $\frac{\gamma'(t)}{\gamma(t)-z_0}$ (as we said before we can reparameterize γ to move at constant rate between vertices, i.e. between t_{i-1}, t_i, γ' is some constant c_i). Then $\phi(t)$ is obviously continuous as a composition of continuous functions ($\gamma(t) - z_0$ is never 0 because $z_0 \notin \mathrm{im}(\gamma)$), and is differentiable at every point of $[a,b] \setminus \{t_0, \dots, t_n\}$ with derivative

$$\phi'(t) = \frac{(\gamma(t)-z_0)I'(t)\exp(I(t)) - \exp(I(t))\gamma'(t)}{(\gamma(t)-z_0)^2} = \frac{\gamma'(t)\exp(I(t)) - \exp(I(t))\gamma'(t)}{(\gamma(t)-z_0)^2} = 0.$$

Having derivative 0 at all but finitely many points means that between those finitely many points ϕ is constant, so ϕ is piecewise constant on $[a,b]$ (with finitely many pieces). In particular the "bad points" t_i are isolated, so at every t_i on sufficiently small neighborhoods the function ϕ attains only some constant C_1 to the left of t_i and some constant C_2 to the right of t_i. As ϕ is continuous at t_i, the left- and right-hand limits must be equal, so $C_1 = C_2$. This applies to every "bad point" t_i, so indeed ϕ is constant on $[a,b]$ and we are done.

Exercise 49

In this exercise, we prove some corollaries of the Jordan curve theorem (Thm. 48 in Notes 3), the statement of which is as follows ($W_\gamma(z_0)$ refers to the winding number $W_\gamma(z_0) := \frac{1}{2\pi i}\oint_\gamma \frac{1}{z-z_0}\,dz$):

> *Theorem: Jordan curve theorem (JCT)*
>
> For a non-trivial simple closed curve $\gamma = [a,b] \to \mathbb{C}$, the open set $\mathbb{C} \setminus \mathrm{im}(\gamma)$ is the disjoint union of two non-empty open connected sets, an "interior region" $\mathrm{int}(\gamma)$ (which is bounded and given by $\mathrm{int}(\gamma) = \{z_0 \in \mathrm{im}(\gamma) : W_\gamma(z_0) = \sigma\}$ where $\sigma \in \{-1, 1\}$ is a constant intrinsic to the path γ known as its "orientation") and an "exterior region" $\mathrm{ext}(\gamma)$ (which is unbounded and given by $\mathrm{ext}(\gamma) = \{z_0 \in \mathrm{im}(\gamma) : W_\gamma(z_0) = 0\}$.

In the following, let γ_1, γ_2 be non-trivial simple closed curves.

(i) If γ_1, γ_2 have disjoint images, we want to show that γ_2 either lies entirely in the interior of γ_1, or entirely in the exterior. *Proof:* suppose that γ_2 does not lie entirely in the interior or exterior of γ_1, meaning that there are $z_0, z_1 \in \mathrm{im}(\gamma_2)$ s.t. $z_0 \in \mathrm{int}(\gamma_1)$ and $z_1 \in \mathrm{ext}(\gamma_1)$. Let $t_0, t_1 \in [a,b]$ be the times corresponding to z_0, z_1 and suppose w.l.o.g. that $t_0 < t_1$.

Then, $(\gamma_2)|_{[t_0,t_1]}$ is a continuous path between z_0, z_1 that lies completely in $\mathbb{C} \setminus \text{im}(\gamma_1)$ (since we assumed γ_1, γ_2 have disjoint images). Thus $z_0 \in \text{int}(\gamma_1), z_1 \in \text{ext}(\gamma_1)$ must lie in the same connected component of $\mathbb{C} \setminus \text{im}(\gamma_1)$; this obviously contradicts the JCT which says that there are only 2 such connected components, the interior and exterior, and they are NOT connected to each other.

(ii) If γ_2 avoids the exterior of γ_1 (so i.e. $\text{im}(\gamma_2) \subseteq \text{im}(\gamma_1) \cup \text{int}(\gamma_1)$ since $\mathbb{C}$ is the disjoint union of $\text{int}(\gamma_1) \sqcup \text{im}(\gamma_1) \sqcup \text{ext}(\gamma_1)$), we want to show that the $\text{int}(\gamma_2) \subseteq \text{int}(\gamma_1)$ and $\text{ext}(\gamma_2) \supseteq \text{ext}(\gamma_1)$. *Proof:* the latter claim follows from the former since $\text{im}(\gamma_2) \subseteq \text{im}(\gamma_1) \cup \text{int}(\gamma_1)$ together with $\text{int}(\gamma_2) \subseteq \text{int}(\gamma_1) \implies \text{int}(\gamma_2) \cup \text{im}(\gamma_2) \subseteq \text{im}(\gamma_1) \cup \text{int}(\gamma_1) \implies \text{ext}(\gamma_2) \supseteq \text{ext}(\gamma_1)$ by taking complements.

The other claim is much harder; see the *Remark* below. But if we assume the fact $\partial\, \text{ext}(\gamma_1) = \text{im}(\gamma_1)$ (there's a very clever proof of this using Brouwer's fixed point theorem and the Tietze extension theorem; see the last section of my undergrad senior thesis), it becomes easy. Let $z_0 \in \text{int}(\gamma_2)$. First, the easy case: suppose f.s.o.c. that $z_0 \in \text{ext}(\gamma_1)$. Then let z_∞ be point very far away, in exterior of both curves (curves and their interiors bounded so we can do this). Then z_0, z_∞ both in $\text{ext}(\gamma_1)$, so they are connected by some path ρ. As $\text{im}(\gamma_2) \subseteq \text{im}(\gamma_1) \cup \text{int}(\gamma_1) = \text{ext}(\gamma_1)^{\complement}$, the path ρ does not intersect γ_2, so ρ connects $z_0 \in \text{int}(\gamma_2)$ and $z_\infty \in \text{ext}(\gamma_2)$; contradiction.

Note that the above proves that $\text{ext}(\gamma_1) \cap \text{int}(\gamma_2) = \varnothing \implies \text{ext}(\gamma_1) \subseteq \text{im}(\gamma_2) \cup \text{ext}(\gamma_2)$. The harder case: suppose f.s.o.c. that $z_0 \in \text{im}(\gamma_1)$. Since $z_0 \in \text{int}(\gamma_2)$ and that set is open, $D(z_0, r) \subseteq \text{int}(\gamma_2)$ for some $r > 0$. Using the fact $\partial\, \text{ext}(\gamma_1) = \text{im}(\gamma_1)$, we get that $D(z_0, r) \cap \text{ext}(\gamma_1)$ is non-empty. But $D(z_0, r) \cap \text{ext}(\gamma_1) \subseteq \text{int}(\gamma_2) \cap [\text{im}(\gamma_2) \cup \text{ext}(\gamma_2)] = \varnothing$; contradiction.

Remark 11/1/21: note that the claim that $\text{im}(\gamma_2) \subseteq \text{im}(\gamma_1) \cup \text{int}(\gamma_1) \implies \text{int}(\gamma_2) \subseteq \text{int}(\gamma_2)$ implicitly rules out the possibility that e.g. the set $S := \partial D(0,1) \cup [0,1]$ could be the image of a simple closed curve γ_1 (the JCT as stated above is not strong enough to rule this possibility out, as indeed $S^{\complement}$ is the disjoint union of two connected open sets $D(0,1) \setminus [0,1) =: \text{int}(S)$ and $\overline{D(0,1)}^{\complement} =: \text{ext}(S)$). This is because if it were that $\text{im}(\gamma_1) = S$ for some γ_1, then setting γ_2 to be the standard parameterization of the circle $\partial D(0, \frac{1}{2})$, we would have $0 \in \text{int}(\gamma_2) = D(0, \frac{1}{2})$ but $0 \notin \text{int}(\gamma_1)$, contradicting the desired claim (the precondition $\text{im}(\gamma_2) \subseteq \text{im}(\gamma_1) \cup \text{int}(\gamma_1) = \overline{D(0,1)}$ is satisfied, but the conclusion $\text{int}(\gamma_2) \subseteq \text{int}(\gamma_2)$ is not reached). This tells us that any proof of the desired claim must implicitly rule out this possibility (as well as e.g. similar possibilities that $\text{im}(\gamma_1)$ is "space-filling"), so all proofs of the desired claim would probably either use tools (like the fact $\partial\, \text{ext}(\gamma) = \text{im}(\gamma)$ that we used), be very long, or be very clever (i.e. using something fundamental about simple closed curves that would rule out the above possibility).

(iii) If γ_2 avoids the interior of γ_1 and γ_1 avoids the interior of γ_2, and the two curves have disjoint images, we want to show that $\text{int}(\gamma_2) \subseteq \text{ext}(\gamma_1)$ and vice versa. *Proof:* from part (i) we know that disjoint images means that either $\text{im}(\gamma_2) \subseteq \text{int}(\gamma_1)$ or $\text{im}(\gamma_2) \subseteq \text{ext}(\gamma_1)$. But we said that γ_2 avoids the interior of γ_1, so we must have $\text{im}(\gamma_2) \subseteq \text{ext}(\gamma_1)$. Suppose that there is $z \in \text{int}(\gamma_2)$ not in $\text{ext}(\gamma_1)$; then since $\mathbb{C}$ is the disjoint union of $\text{int}(\gamma_1) \sqcup \text{im}(\gamma_1) \sqcup \text{ext}(\gamma_1)$, z must be either in $\text{im}(\gamma_1)$ or $\text{int}(\gamma_1)$.

However, z can not be in $\text{im}(\gamma_1)$, because we said that γ_1 avoids the interior of γ_2 (and z was original chosen to be in $\text{int}(\gamma_2)$). Thus, $z \in \text{int}(\gamma_2) \cap \text{int}(\gamma_1)$. Let us now pick some extremely far-out point, z_∞ in the exterior of both γ_1, γ_2, and draw the line $[z \to z_\infty]$. As z is in the interior of both curves and z_∞ is in the exterior of both curves, by a connectedness argument (I did the same thing in part (i) above), the line $[z \to z_\infty]$ must intersect both $\text{im}(\gamma_1)$ and $\text{im}(\gamma_2)$.

Let z_1 be the first intersection point between the ray $[z \to z_\infty]$ and the curves $\gamma_{1,2}$ (the intersection of two compact sets is compact and in particular closed, i.e. containing all its limit points, so there is such a thing as "the first intersection"), and say it's on $\text{im}(\gamma_i)$ (for $i \in \{1,2\}$). We can't have z_1 on both curves since γ_1, γ_2 have disjoint images, so indeed z_1 must be in the interior of γ_j (for $i \neq j \in \{1,2\}$), since otherwise $z_1 \in \text{ext}(\gamma_1)$ meaning by the same connectedness there must have been a previous time at which the ray $[z \to z_\infty]$ intersected $\text{im}(\gamma_1)$; but this is impossible since we defined z_1 to be the *first* intersection between the ray $[z \to z_\infty]$ and the curves γ_1, γ_2.

But of course $z_1 \in \text{im}(\gamma_2) \cap \text{int}(\gamma_1)$ is impossible since we said that γ_2 avoids the interior of γ_1! So indeed all paths lead to contradiction, so we must have $\text{int}(\gamma_2) \subseteq \text{ext}(\gamma_1)$. The "vice versa" follows by symmetry (the situation between γ_1, γ_2 is symmetrical so we need only swap every "1" and "2" in the above proof).

246A Homework 4

Daniel Rui - 10/29/21

Exercise 3(iii) (Notes 3)

If $\gamma_0 : [a,b] \to U, \gamma_1 : [c,d] \to U$ are closed curves with the same initial point z_0 (and hence same terminal point), we want to show that γ_0 is homotopic as closed curves (h.a.c.c.) to γ_1 up to reparameterization (u.t.r) if and only if γ_0 is homotopic with fixed endpoints (h.w.f.e) to $\gamma_2 + \gamma_1 + (-\gamma_2)$ u.t.r. for some closed curve γ_2 with the same initial (and terminal) point as γ_0 and γ_1 (i.e. for closed curves with the same endpoints, "homotopic as closed curves" $\iff$ "homotopic with fixed endpoints" up to conjugation).

Without loss of generality, we may take γ_0, γ_1 to be defined on $[a,b]$ and to be actually homotopic as closed curves, since by Def. 1(iii) in Notes 3 we can simply instead consider reparameterizations $\tilde{\gamma}_0 \equiv \gamma_0, \tilde{\gamma}_1 \equiv \gamma_1$ (using the notation introduced in the paragraph before Ex. 2 in Notes 2), and then at the end when we get $\tilde{\gamma}_0$ is h.w.f.e. to $\gamma_2 + \tilde{\gamma}_1 + (-\gamma_2)$ (perhaps u.t.r.), that will imply γ_0 is h.w.f.e. to $\gamma_2 + \gamma_1 + (-\gamma_2)$ u.t.r., where we are taking advantage of the fact that concatenation is "well-defined up to equivalence" by Ex. 3(i) in Notes 2, or more specifically in our case $\gamma_2 + \tilde{\gamma}_1 + (-\gamma_2) \equiv \gamma_2 + \gamma_1 + (-\gamma_2)$.

So now γ_0 is h.a.c.c. to γ_1. That means there is a continuous "deformation map" $\Gamma : [0,1] \times [a,b] \to U$ s.t. $\Gamma(0,t) = \gamma_0(t)$ and $\Gamma(1,t) = \gamma_1(t)$ for all $t \in [a,b]$, and $\Gamma(s,a) = \Gamma(s,b)$ for all $s \in [0,1]$. Let us define continuous $\gamma_2 : [0,1] \to U$ by $\gamma_2(s) := \Gamma(s,a)$ (or equivalently $= \Gamma(s,b)$). Reviewing the definition of concatenation in the paragraph before Ex. 3 in Notes 2, and using the aforementioned fact that concatenation is "well-defined up to equivalence" (Ex. 3(i) in Notes 2), we can think of γ_2 as being defined on $[a-1,a]$ and $(-\gamma_2)$ as being defined on $[b-1,b]$ (i.e. we reparameterize by rigid translation) so that $\gamma_2 + \gamma_1 + (-\gamma_2)$ is a continuous map $[a-1,b+1] \to U$. Note that originally $(-\gamma_2)(t) := \gamma_2(-t)$ for $t \in [-1,0]$ but with these rigid translations we now have for $t \in [0,1]$, $(-\gamma_2)(b+t) := \gamma_2(a-t)$, and of course $\gamma_2((a-1)+s) := \Gamma(s,a) = \Gamma(s,b)$.

Let us now define the "extension" $\hat{\gamma}_0 : [a-1,b+1] \to U$ by $\hat{\gamma}_0([a-1,a]) = \hat{\gamma}_0([b,b+1]) = \{z_0\}$, i.e. $\hat{\gamma}_0$ starts at the initial/terminal point $z_0 := \gamma_0(0) = \gamma_0(1)$, stays there for one unit of time, walks the path γ_0, then stays at the initial/terminal point z_0 again for another unit of time (in particular $\hat{\gamma}_0$ is just a reparameterization, i.e. $\hat{\gamma}_0 \equiv \gamma_0$).

Let us now define the homotopy $\Upsilon : [0,1] \times [a-1,b+1] \to U$ between $\hat{\gamma}_0$ and $\gamma_2 + \gamma_1 + (-\gamma_2)$ by $\Upsilon(s,t) := \Gamma(s,t)$ on $[0,1] \times [a,b]$; and on $[0,1] \times [a-1,a]$, $\Upsilon(s,t' + (a-1)) := \gamma_2((a-1)+st')$, $t' \in [0,1]$ (so for each fixed $s_0 \in [0,1]$, $\Upsilon(s_0,t)$ walks along γ_2 from $\gamma_2(a-1)$ to $\gamma_2((a-1)+s_0) = \Gamma(s_0,a)$ in unit time, by slowing down its walking speed along γ_2 by a factor of $s_0 \in [0,1]$); and similarly on $[0,1] \times [b,b+1]$, $\Upsilon(s,t'+b) := (-\gamma_2)((b+(1-s))+st')$, $t' \in [0,1]$ (so for each fixed $s_0 \in [0,1]$, $\Upsilon(s_0,t)$ walks along $(-\gamma_2)$ from $(-\gamma_2)(b+(1-s_0)) = \gamma_2(a-(1-s_0)) = \gamma_2((1-a)+s_0) = \Gamma(s_0,b)$ to $(-\gamma_2)(b+1) = \gamma_2(a-1)$ in unit time).

Essentially, we're just pulling $\gamma_1(\bullet)$ back along $\gamma_2(\bullet)$ via $\Gamma(s, \bullet)$ to $\gamma_0(\bullet)$, but extending both ends of each intermediate curve $\Gamma(s, \bullet)$ along γ_2 to make sure the intermediate curves all have initial/terminal/end point equal to the initial/terminal/end point of γ_0, so as to make it a "homotopy with fixed endpoints).

Looking at the formulas above, we have defined Υ on $[0,1] \times [a-1, b+1]$ on 3 closed subsets that cover $[0,1] \times [a-1, b+1]$ (well-defined on the overlaps of the closed subsets), and on each of those closed subsets $\Upsilon(s,t)$ is a continuous function (according to the formulas above, $\Upsilon(s,t)$ on any of those 3 closed subsets is a composition of continuous functions γ_2, $(-\gamma_2)$, and linear/affine interpolation functions), and so by the pasting/gluing lemma for closed subsets, $\Upsilon : [0,1] \times [a-1, b+1] \to U$ is continuous. By construction it is a homotopy between $\hat{\gamma}_0$ and $\gamma_2 + \gamma_1 + (-\gamma_2)$, and all intermediate curves have endpoints $\gamma_2(a-1) = \Gamma(0,a) = \Gamma(0,b) = \gamma_0(a) = \gamma_0(b) = z_0$, so we are done.

For the other direction, the h.w.f.e. Γ will continuously deform $\gamma_2 + \gamma_1 + (-\gamma_2)$ to γ_0, so $\gamma_1(a)$ is sent to some $\gamma_0(\alpha)$ and $\gamma_1(b)$ is sent to some $\gamma_0(\beta)$ for $a \leqslant \alpha \leqslant \beta \leqslant b$. This procedure will be the first half of the total homotopy Υ. The second half is as follows: the first half of Υ sent $\gamma_1(t)$ to $\gamma_0((1-w)\alpha + w\beta)$ for some weight $w = w_t \in [0,1]$ (since the homotopy is continuous, w depends continuously on t). Then, the second half of Υ will send $\gamma_0((1-w)\alpha + w\beta)$ to $\gamma_0((1-w)a + wb)$ travelling along γ_0 (more explicitly, we could do a linear interpolation map i.e. for a time parameter $\tau \in [0,1]$, we travel $\gamma_0\big((1-w)[(1-\tau)\alpha + \tau a] + w[(1-\tau)\beta + \tau b]\big)$; in this presentation it is more clearly a continuous map). Together the whole Υ sends $\gamma_1(a)$ to $\gamma_0(a)$ and $\gamma_1(b)$ to $\gamma_0(b)$, and everything "in the middle" $\gamma_1(t)$ continuously to something "in the middle" $\gamma_0((1-w_t)a + w_t b)$; i.e. Υ is our desired homotopy between γ_1 and γ_0 as closed curves.

EDIT 11/20/21: the bit about the weight w_t depending continuously on t is difficult to justify, but there is an easier homotopy, namely if we stretch γ_1 to $\gamma_2 + \gamma_1 + (-\gamma_2)$ and then apply the homotopy Γ to map it to γ_0, as opposed to what we did in the previous paragraph which was to apply Γ then stretch the image of γ_1 (under Γ) to γ_0. Explicitly, "stretching γ_1 to $\gamma_2 + \gamma_1 + (-\gamma_2)$" means we take $a(1-w)+bw$ linearly to $(a-1)(1-w)+(b+1)w$, so $[\gamma_2+\gamma_1+(-\gamma_2)](a(1-w)+bw) = \gamma_1(a(1-w)+bw)$ gets taken to $[\gamma_2 + \gamma_1 + (-\gamma_2)]((a-1)(1-w) + (b+1)w)$.

Exercise 3(iv)

We define a *point curve* in U to be a curve $\gamma_1 : [a,b] \to U$ of the form $\gamma_1(t) = z_0$ for some fixed $z_0 \in U$ and all $t \in [a,b]$. Let $\gamma_0 : [a,b] \to U$ be just any closed curve in U. We want to show that γ_0 is homotopic with fixed endpoints (h.w.f.e) to a point curve in U if and only if γ_0 is homotopic as closed curves (h.a.c.c.) to a point curve in U (in either case, we call γ_0 "homotopic to a point", "null-homotopic", "contractible to a point in U", or just "contractible").

The ($\implies$) direction is trivial because by the definition (Def. 1 in Notes 3) of h.w.f.e. and h.a.c.c., any γ_0, γ_1 that have the same initial/terminal/end point and are h.w.f.e. have homotopy γ satisfying $\gamma(s,a) = z_0 = \gamma(s,b)$, where in particular $\gamma(s,a) = \gamma(s,b)$ shows that γ_0, γ_1 are h.a.c.c. via γ.

For the other ($\Longleftarrow$) direction, given a homotopy Γ continuously deforming γ_0 (a closed curve with initial/terminal/end point $= z_0$) to some point curve $z_1 \in U$, define $\gamma_1(s)$ to be the curve $\Gamma(s,a) = \Gamma(s,b)$ (for $s \in [0,1]$) going from z_0 to z_1, i.e. the path of the endpoints of the intermediate curves $\Gamma(s,\bullet)$ (γ_1 is a continuous curve in U because Γ is continuous map into U). Then we can construct a homotopy between γ_0 and z_0, denoted by Υ, with fixed initial/terminal/end point $= z_0$ as follows: for every $s \in [0,1]$, define the intermediate curve $\Upsilon(s,\bullet)$ to go from z_0 to $\Gamma(s,a) = \Gamma(s,b)$ via γ_1, then walk around $\Gamma(s,\bullet)$ back to $\Gamma(s,a) = \Gamma(s,b)$, then back to z_0 via γ_1; at $s = 1$, $\Upsilon(s,\bullet)$ will be precisely $\gamma_1 + (-\gamma_1)$, at which point we can define $\Upsilon(s,\bullet)$ for $s \in [1,2]$ say to be the "retraction" along γ_1 back to z_0, i.e. $\Upsilon(s,\bullet) = \gamma_1|_{[0,s]} + (-\gamma_1|_{[0,s]})$.

The explicit formulas for the above procedure are much the same as the formulas above in Ex. 3(iii) (in fact in that problem I already introduced language such as "walk along a curve" or "go from one point to another along a curve").

Exercise 6

We are asked to show that the following two forms of Cauchy's theorem are equivalent. Let $U \subseteq \mathbb{C}$ be open, and let $f : U \to \mathbb{C}$ be holomorphic. We use the above abbreviations — homotopic as closed curves (h.a.c.c.); up to reparameterization (u.t.r); and homotopic with fixed endpoints (h.w.f.e).

- (1) If γ_0, γ_1 are rectifiable curves that are either h.w.f.e. in U u.t.r. or h.a.c.c. in U u.t.r., then $\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz$.
- (2) If γ is a rectifiable closed curve in U contractible to a point, then $\int_\gamma f(z)\,dz = 0$.

As noted in Notes 3, (1) trivially implies (2), so we focus on (2) $\implies$ (1). Given $\tilde{\gamma}_0, \tilde{\gamma}_1$ rectifiable curves that are h.w.f.e./h.a.c.c. in U u.t.r., we have $\tilde{\gamma}_0 \equiv \gamma_0, \tilde{\gamma}_1 \equiv \gamma_1$ for reparameterizations $\gamma_0, \gamma_1 : [a,b] \to U$ that are h.w.f.e./h.a.c.c. in U — in particular, since integrals are invariant under curve reparameterization, it suffices to show $\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz$.

The rest boils down to two observations. (1) For γ_0, γ_1 rectifiable and h.w.f.e. in U via the homotopy Γ (let z_0 be the initial point of both curves), we may form a homotopy Υ that first continuously deforms the curve $\gamma_0 + (-\gamma_1)$ to $\gamma_1 + (-\gamma_1)$ via Γ (essentially this first half of Υ does not change the $(-\gamma_1)$ portion of the curve, and moves $\gamma_0(\bullet)$ along the intermediate paths $\Gamma(s,\bullet)$ continuously until $\gamma_0(\bullet)$ has deformed into $\gamma_1(\bullet)$); then Υ will retract along γ_1 back to z_0 (in the sense defined in the third paragraph of Ex. 4 above). Then, Υ is a homotopy between $\gamma_0 + (-\gamma_1)$ and the point curve $[z_0]$, so $\int_{\gamma_0} - \int_{\gamma_1} = \int_{\gamma_0} + \int_{(-\gamma_1)} = \oint_{\gamma_0 + (-\gamma_1)} = \oint_{[z_0]} = 0$, giving us $\int_{\gamma_0} = \int_{\gamma_1}$ as desired.

(2) Similarly, for γ_0, γ_1 rectifiable and h.a.c.c. in U via the homotopy Γ (let $\gamma_2(s) = \Gamma(s,a)$ be the trajectory of the endpoint of the intermediate closed curves), we may form a homotopy Υ that first continuously deforms the curve $\gamma_0 + \gamma_2 + (-\gamma_1) + (-\gamma_2)$ (closed curve with endpoint $z_0 := \gamma_0(a)$) along the intermediate curves $\gamma_0|_{[0,t]}(\bullet) + \Gamma(\bullet, t) + (-\gamma_1|_{[0,t]})(\bullet) + (-\gamma_2)(\bullet)$ (closed curves with endpoint z_0) until it reaches $\gamma_2 + (-\gamma_2)$, at which point (like in the above case) Υ retracts

along γ_2 back to z_0 (in the sense defined in the third paragraph of Ex. 4 above). The deformation along the intermediate curves is continuous because $\gamma|_{[0,t]}$ and $\Gamma(\bullet, t)$ change continuously in $t \in [a, b]$. Then, Υ is a homotopy between $\gamma_0 + \gamma_2 + (-\gamma_1) + (-\gamma_2)$ and the point curve $[z_0]$, so $\int_{\gamma_0} - \int_{\gamma_1} = \int_{\gamma_0} + \int_{\gamma_2} - \int_{\gamma_1} - \int_{\gamma_2} = \int_{\gamma_0} + \int_{\gamma_2} + \int_{(-\gamma_1)} + \int_{(-\gamma_2)} = \oint_{\gamma_0+\gamma_2+(-\gamma_1)+(-\gamma_2)} = \oint_{[z_0]} = 0$, giving us $\int_{\gamma_0} = \int_{\gamma_1}$ as desired.

Another route is simply the way Terry proved Cauchy's theorem from local Cauchy's theorem (see paragraphs after Corollary 13 in Notes 3), i.e. knowing that for any closed curve γ in some open ball B on which f is holomorphic (in particular γ would be contractible in B via a straight-line homotopy by convexity of B) the integral $\oint_\gamma f(z)\,dz$ is equal to 0, we can prove general Cauchy's theorem (1) by putting a fine enough mesh on $[0, 1] \times [a, b]$ so that the homotopy Γ maps it to a fine enough "grid" of (closed) quadrilaterals, where each quadrilateral is small enough that it's contained in some open ball $B \subseteq U$, thereby allowing us to apply local Cauchy's theorem to get that the integral $\oint$ over each tiny (closed) quadrilateral is 0. More details again found in paragraphs after Cor. 13 in Notes 3.

Exercise 12

We are asked to use the (real-variable) FTC(Id) and Fubini's theorem in place of Goursat's theorem to prove Cauchy's theorem for rectangles $\gamma_R := [x_0 + iy_0 \to x_1 + iy_0 \to x_1 + iy_1 \to x_0 + iy_1 \to x_0 + iy_0]$, **allowing the further assumption** that f' is continuous (i.e. f is $\mathcal{C}^1$ on U, and $R \subseteq U$).

We prove Green's theorem for rectangles directly. The statement of Green's theorem for such a rectangle is as follows: for $R = \{(x, y) \subseteq \mathbb{R}^2 : x \in (x_0, x_1), y \in (y_0, y_1)\}$, we have that for $P, Q \in \mathcal{C}^1$ on $\overline{R} = R \cup \partial R$ (i.e. there is some open set containing $\overline{R}$ on which P, Q are defined and $\mathcal{C}^1$), then the two integrals

$$\begin{aligned}\int_{\partial R} P\,dx + Q\,dy &:= \int_{x_0}^{x_1} P(x, y_0)\,dx + \int_{y_0}^{y_1} Q(x_1, y)\,dy + \int_{x_1}^{x_0} P(x, y_1)\,dx + \int_{y_1}^{y_0} Q(x_0, y)\,dy \\ &= \int_{x_0}^{x_1} P(x, y_0) - P(x, y_1)\,dx + \int_{y_0}^{y_1} Q(x_1, y) - Q(x_0, y)\,dy\end{aligned}$$

(where the path we took for ∂R started at (x_0, y_0) and walked counterclockwise around) and

$$\iint_R (\partial_x Q - \partial_y P)\,dx\,dy = \int_{y_0}^{y_1} \int_{x_0}^{x_1} (\partial_x Q - \partial_y P)\,dx\,dy$$

are equal. Because Q, P are $\mathcal{C}^1$ on $\overline{R}$, the first partials are continuous on $\overline{R}$, hence L^1-integrable (continuous on a compact set implies bounded), Fubini and the "undergraduate" fundamental theorem of calculus gives that

$$\begin{aligned}\int_{y_0}^{y_1} \int_{x_0}^{x_1} (\partial_x Q - \partial_y P)\,dx\,dy &= \int_{y_0}^{y_1} \int_{x_0}^{x_1} \partial_x Q\,dx\,dy - \int_{x_0}^{x_1} \int_{y_0}^{y_1} \partial_y P\,dy\,dx \\ &= \int_{y_0}^{y_1} Q(x_1, y) - Q(x_0, y)\,dy - \int_{x_0}^{x_1} P(x, y_1) - P(x, y_0)\,dx,\end{aligned}$$

which is exactly the expression we found above for $\int_{\partial R} P\,dx + Q\,dy$.

We now apply this result to the complex setting. Note that the above red line is precisely the definition of the complex line integral $\oint_{\gamma_R} f(z)\,dz$, replacing P with f and Q with if (one can see this via the "change of variables formula" which we proved last week in Ex.17(iv) from Notes 2, applied to the 4 paths that walk from each corner to the next counterclockwise at "unit speed", i.e. such that $\gamma'(t)$ on each respective path is $1, i, -1, -i$). So Green's theorem as proven above says that $\oint_{\gamma_R} f(z)\,dz = \int_{\partial R} f\,dx + if\,dy = \int_{y_0}^{y_1}\int_{x_0}^{x_1}(i\partial_x f - \partial_y f)\,dx\,dy$. But the Cauchy-Riemann equations tells us that f holomorphic on $U \implies \partial_y f = i\partial_x f$ on U! So the RHS integrand is identically 0, so indeed $\oint_{\gamma_R} f(z)\,dz = 0$, as desired.

Exercise 17

Let $U \subseteq \mathbb{C}$ be open and let $\overline{D(z_0, r)} \subseteq U$ be any closed disk contained in U. This exercise is about harmonic functions (i.e. $\mathcal{C}^2$ functions $u : U \to \mathbb{R}$ satisfying $\Delta u := [\partial_x u]^2 + [\partial_y u]^2 = 0$; see the paragraph before Thm. 25 in Notes 1), the mean value property, and the Poisson kernel.

(i) For holomorphic $f : U \to \mathbb{C}$, we want to show that $f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta$, and then use this "mean value property" to (re)prove the maximum principle for holomorphic functions (see Ex. 26 from Notes 1). Well, the most basic form of the Cauchy integral formula (Thm. 15 in Notes 3) says that since we have $\overline{D(z_0, r)} \subseteq U$, $f(z_0) = \frac{1}{2\pi i}\oint_{\gamma_{z_0,r,\circlearrowleft}} \frac{f(z)}{z-z_0}\,dz$, where we parameterize $\gamma_{z_0,r,\circlearrowleft}$ as $\gamma(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$. This is in particular a $\mathcal{C}^1$ curve, allowing us to use the "change of variables formula" $\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt$ (which we proved last week in Ex.17(iv) from Notes 2), as follows:

$$f(z_0) = \frac{1}{2\pi i}\oint_{\gamma_{z_0,r,\circlearrowleft}} \frac{f(z)}{z - z_0}\,dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{it})}{re^{it}}(ire^{it})\,dt = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\,dt.$$

As promised, we now use this to prove the maximum principle for holomorphic functions (i.e. the maximum modulus principle). Let $K \subseteq U$ be compact, and let ∂K be the boundary of K. Because $|f|$ is a continuous function, it attains its supremum/maximum M on K, say $|f|(z_0) = M$ for $z_0 \in K$. Suppose now f.s.o.c. that $|f|$ never attains M on ∂K, so z_0 must be in the interior $\mathrm{int}(K)$ of K.

Let O be the open (path-)connected component of $\mathrm{int}(K)$ that z_0 resides in. Then $\partial O \subseteq \overline{O} \subseteq K$, and $z \in \partial O$ can not be in the interior of K (otherwise there is a small ball around z contained completely in some open (path-)connected component of $\mathrm{int}(K)$, i.e. either completely in O or completely outside O, contradicting that z is on the boundary of O), implying that $\partial O \subseteq \partial K$. Thus, it is enough (to reach contradiction) to show that $|f|$ attains the maximal value on M on ∂O.

First, we show that $|f|$ attains M on all of O. We do this using the "continuity method", i.e. using the connectedness of O to argue that any non-empty set S that is both closed and open in O must be all of O. Define S to be the set of $z \in O$ s.t. $|f|(z) = M$. Clearly $S \neq \varnothing$ because

$z_0 \in S$. It is closed because $|f|$ is continuous and the inverse image of the closed set $\{M\}$ via the continuous map $|f|$ is closed in O. Now we show it's open. Suppose $z_1 \in S$. Then since O is open, there is $\delta > 0$ s.t. $r \in (0,\delta) \implies \overline{D(z_1,r)} \subseteq O$. Then, $M - |f|(z_1 + re^{it})$ is a continuous function of $t \in [0,2\pi]$ that is $\geqslant 0$ (since M is maximal value). Moreover, this continuous function on $[0,2\pi]$ has an integral of 0: $M = |f|(z_1) \leqslant \frac{1}{2\pi}\int_0^{2\pi} |f|(z_1 + re^{it})\,dt \leqslant \frac{1}{2\pi}\int_0^{2\pi} M\,dt = M \implies \frac{1}{2\pi}\int_0^{2\pi} |f|(z_0) - |f|(z_1 + re^{it})\,dt = \frac{1}{2\pi}\int_0^{2\pi} M\,dt - \frac{1}{2\pi}\int_0^{2\pi} |f|(z_1 + re^{it})\,dt = 0$.

The only continuous function that has integral 0 over a non-trivial interval is precisely the 0 function, so indeed $|f|(z_1 + re^{it}) = |f|(z_1) = M$. But $r \in (0,\delta)$ was arbitrary, so we have just shown that $|f| = M$ on all of $D(z_1,\delta)$. As $z_1 \in O$ too was arbitrary, S is open in O, and by connectedness (as advertised) $S = O$. Finally, because $|f|$ is continuous on U, in particular on $K \subseteq U$, $|f|^{-1}(\{M\})$ is a closed set in K containing O, so in particular must contain the smallest closed set containing O, i.e. $\overline{O}$. We have therefore shown that $|f|$ attains its maximum M on $\partial O \subseteq \partial K$, contradicting our initial assumption that $|f| \neq M$ anywhere on the boundary ∂K.

(ii) We are asked to do the exact same thing as in part (i) except replace "holomorphic $f : U \to \mathbb{C}$" with "harmonic $u : U \to \mathbb{R}$". Observe that once we establish the mean value property for harmonic functions, the maximum principle for harmonic functions (cf. Thm. 25 of Notes 1) follows immediately by replacing "$|f|$" in the second half of part (i) above with "u" (yes, word for word, everything still works perfectly fine).

Recall from Prop. 28 in Notes 1 that any harmonic function $u : \mathbb{C} \to \mathbb{R}$ has a harmonic conjugate $v : \mathbb{C} \to \mathbb{R}$ (i.e. $u + iv$ is holomorphic) given by the formula $v(x+iy) := -\int_0^x [\partial_y u](t)\,dt + \int_0^y [\partial_x u](x+it)\,dt$. The exact proof given in the aforementioned Prop. 28, centered at arbitrary $z_0 \in U$ instead of 0, holds for any open $U \subseteq \mathbb{C}$ that contains the line segments $[z_0 \to x + \mathfrak{Im}(z_0)]$ and $[x + \mathfrak{Im}(z_0) \to x + iy]$ for any $x + iy \in U$. In particular for any disk $D(z_0,r)$ on which u is harmonic, the exact proof of Prop. 28 (again centered at z_0 instead of at 0, i.e. now $v(x+iy) := -\int_{\mathfrak{Re}(z_0)}^x [\partial_y u](t + \mathfrak{Im}(z_0))\,dt + \int_{\mathfrak{Im}(z_0)}^y [\partial_x u](x+it)\,dt$) gives a harmonic conjugate $v : D(z_0,r) \to \mathbb{R}$ s.t. $u + iv$ is holomorphic on $D(z_0,r)$.

Now suppose we have some arbitrary closed disk $\overline{D(z_0,r)} \subseteq U$ for some open $U \subseteq \mathbb{C}$. Because disjoint compact/closed sets are distant, there is some positive $d > 0$ s.t. $z \in \overline{D(z_0,r)}, w \in U^{\complement} \implies |z - w| \geqslant d$, and so $\overline{D(z_0,r)} \subseteq D(z_0, r + \frac{d}{2}) \subseteq U$. Now we have a disk $D(z_0, r + \frac{d}{2})$ on which $u : U \to \mathbb{R}$ is harmonic, so by the above paragraph we have some harmonic conjugate $v : D(z_0, r + \frac{d}{2}) \to \mathbb{R}$ s.t. $u + iv$ is holomorphic on that disk. Applying part (i) (i.e. the mean value property for holomorphic functions) on $\overline{D(z_0,r)} \subseteq D(z_0, r + \frac{d}{2})$ we get $[u+iv](z_0) = \frac{1}{2\pi}\int_0^{2\pi} [u+iv](z_0 + re^{i\theta})\,d\theta$, and comparing real parts we get $u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\,d\theta$ as desired.

(iii) For $u : U \to \mathbb{R}$ harmonic and $\overline{D(z_0,r)} \subseteq U$, we want to prove the Poisson integral formula, namely $u(z) = \frac{1}{2\pi}\int_0^{2\pi} P(\frac{z - z_0}{re^{i\theta}}) u(z_0 + re^{i\theta})\,d\theta$ for any $z \in D(z_0,r)$ and the Poisson kernel $P(z) := \mathfrak{Re}(\frac{1+z}{1-z}) : D(0,1) \to \mathbb{R}$. EDIT 11/15/21 (up until "OLD" section): As the hint suggests, we simplify the problem by transforming the "frame of reference" to set $z_0 = 0, r = 1, z = s$ for some $s \in (0,1) \subseteq \mathbb{R}$. Let us use the notation "z_1" in place of "z" to emphasize we have fixed it a priori. Define the transformation map $\phi : \mathbb{C} \to \mathbb{C}$ by $\phi(w) = \frac{w-z_0}{r\omega}$ where $\omega := \frac{z_1 - z_0}{|z_1 - z_0|} =: e^{i\theta_1}$ (intuitively, ω measure the angle z_1 is at w.r.t. z_0, so dividing by ω rotates our "frame of reference" s.t. z_1 has 0 rotation) and define $\tilde{u}(w) := u(\phi^{-1}(w))$ so that $\tilde{u}(\phi(w)) = u(w)$.

In particular this gives $u(z_1) = \tilde{u}(\frac{z_1-z_0}{r\omega}) = \tilde{u}(\frac{|z_1-z_0|}{r})$ (where $s := \frac{|z_1-z_0|}{r} \in (0,1)$ for $z_1 \in D(z_0, r) \setminus \{z_0\}$; for $z_1 = z_0$ the claim is merely the mean value property we did earlier in part (i)), and $u(z_0 + re^{i\theta}) = \tilde{u}(e^{i(\theta-\theta_1)})$. And of course $P(\frac{z_1-z_0}{re^{i\theta}}) = P(\frac{z_1-z_0}{|z_1-z_0|}\frac{|z_1-z_0|}{re^{i\theta}}) = P(se^{-i(\theta-\theta_1)})$. And as $e^{i\theta}$ is 2π-periodic and the integral is taken from 0 to 2π, we have that $\int_0^{2\pi} P(se^{-i(\theta-\theta_1)})\tilde{u}(e^{i(\theta-\theta_1)})\, d\theta = \int_0^{2\pi} P(se^{-i\theta})\tilde{u}(e^{i\theta})\, d\theta$. So indeed proving that $\tilde{u}(s) = \frac{1}{2\pi}\int_0^{2\pi} P(se^{-i\theta})\tilde{u}(e^{i\theta})\, d\theta$ is equivalent to proving that $u(z_1) = \frac{1}{2\pi}\int_0^{2\pi} P(\frac{z_1-z_0}{re^{i\theta}})u(z_0 + re^{i\theta})\, d\theta$. We write "$u$" instead of "$\tilde{u}$" for convenience (we will not talk about the original u again). Piggybacking off of below justifications, we have

$$f(z) = \frac{1}{2\pi i}\oint_{\gamma_{0,1,\circlearrowleft}} \frac{f(w)}{w-s}\, dw = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(e^{i\theta})}{e^{i\theta}-s} \cdot (ie^{i\theta})\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta}) \cdot \frac{e^{i\theta}}{e^{i\theta}-s}\, d\theta.$$

To pull of the concluding manipulation

$$\begin{aligned} f(z) - 0 &= \frac{1}{2\pi i}\oint_{\gamma_{0,1,\circlearrowleft}} \frac{f(w)}{w-s}\, dw - \frac{1}{2\pi i}\oint_{\gamma_{0,1,\circlearrowleft}} \frac{f(w)}{w-s^{-1}}\, dw \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta}) \cdot \left(\frac{e^{i\theta}}{e^{i\theta}-s} - \frac{e^{i\theta}}{e^{i\theta}-s^{-1}}\right) d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta}) \cdot \mathfrak{Re}\left(\frac{e^{i\theta}+s}{e^{i\theta}-s}\right) d\theta. \end{aligned}$$

(from which we take real parts to recover the claim for harmonic u instead of holomorphic f), $\mathfrak{Re}\left(\frac{e^{i\theta}+s}{e^{i\theta}-s}\right) = \frac{e^{i\theta}}{e^{i\theta}-s} - \frac{e^{i\theta}}{e^{i\theta}-s^{-1}}$, which also requires some messy identity about complex numbers like the one in turquoise in the "OLD" section below. I guess the extra transformation did not make things that much easier. Terry also suggested considering $\frac{1}{2\pi i}\oint_{\gamma_{0,1,\circlearrowleft}} f(w) \cdot \frac{1}{2}(\frac{1+s/w}{1-s/w} + \frac{1+sw}{1-sw}) \cdot \frac{1}{w}\, dw$, but at this point I'm pretty sure either way this problem is just a lot of messy computational details.

EDIT 11/20/21: for the sake of finding closure for myself, I spell out some final computations: $\frac{1+s/w}{1-s/w} + \frac{1+sw}{1-sw} = \frac{w+s}{w-s} + \frac{s^{-1}+w}{s^{-1}-w} = \frac{w-s+2s}{w-s} - \frac{w-s^{-1}+2s^{-1}}{w-s^{-1}} = 1 + \frac{2s}{w-s} - 1 - \frac{2s^{-1}}{w-s^{-1}}$ and the below identity $\overline{(\frac{a+b}{a-b})} = \frac{\overline{a+b}}{\overline{(a-b)}}$ gives $\overline{(\frac{w+s}{w-s})} = \frac{\overline{w+s}}{\overline{(w-s)}} = \frac{\overline{w}+s}{\overline{w}-s} \cdot \frac{w}{w} = \frac{1+sw}{1-sw}$ so indeed $\frac{1+s/w}{1-s/w} + \frac{1+sw}{1-sw} = 2\,\mathfrak{Re}(\frac{w+s}{w-s})$. The $\frac{1}{w}$ at the end is so that the $\gamma'(t)$ factor from the change of variables Ex. 17(iv), namely $ie^{i\theta}$, cancels with $\frac{1}{w} = \frac{1}{e^{i\theta}}$. Finally note that $\frac{w}{w-s} = \frac{w-s+s}{w-s} = \frac{s}{w-s}$, so everything works out. Again, as I correctly intuited before, this does not cut down substantially on the computations.

OLD, because I was stupid and couldn't think of the variable substitution

As the hint suggests, we simplify the problem a tiny bit by translating the "frame of reference" to set $z_0 = 0$ (we do not do as the hint suggests and rotate or scale, though). So now the problem is to prove $u(z) = \frac{1}{2\pi}\int_0^{2\pi} P(\frac{z}{re^{i\theta}})u(re^{i\theta})\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} \mathfrak{Re}(\frac{re^{i\theta}+z}{re^{i\theta}-z})u(re^{i\theta})\, d\theta$ for any $z \in D(0,r)$.

We use the trick from part (ii) to define a holomorphic $f := u + iv$ on a slight enlargement $D(0, r')$ of $\overline{D(0,r)} \subseteq D(0, r') \subseteq U$. We prove the Poisson integral formula for f, and the claim for u will pop right out by taking real parts. By the Cauchy integral formula (Thm. 15 in Notes 3 — $\gamma_{z,\epsilon,\circlearrowleft}$ homotopic to $\gamma_{0,r,\circlearrowleft}$ in $D(0,r') \setminus \{0\}$ via straight-line/weighted-average homotopy) and the "change of variables formula" (Homework 3, Ex.17(iv) from Notes 2 — standard parameterization of circular paths is $\mathcal{C}^1$) we have

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_{0,r,\circlearrowleft}} \frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(re^{i\theta})}{re^{i\theta}-z}\cdot(ire^{i\theta})\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})\cdot\frac{re^{i\theta}}{re^{i\theta}-z}\,d\theta.$$

On the other hand, note the following interesting facts: defining $z' := \frac{r^2}{|z|^2}z$, we have that $|z| < r \implies r|z| < r^2 \implies |z'| = \frac{r^2}{|z|} > r \implies z'$ is outside $\overline{D(z_0,r)}$, so $\frac{f(w)}{w-z'}$ is holomorphic as a function of w on a small enough enlargement $D(z_0, r'')$ of $\overline{D(z_0,r)} \subseteq D(z_0,r'') \subseteq D(z_0,r') \subseteq U$, hence by Cauchy's theorem $\frac{1}{2\pi i}\int_{\gamma_{0,r,\circlearrowleft}} \frac{f(w)}{w-z'}\,dw = 0$; and $\frac{re^{i\theta}}{re^{i\theta}-z'} = \frac{|z|^2 re^{i\theta}}{|z|^2re^{i\theta}-r^2z} = \frac{\overline{z}e^{i\theta}}{\overline{z}e^{i\theta}-r} = \frac{\overline{z}}{\overline{z}-re^{-i\theta}} = \frac{\overline{z}}{\overline{(z-re^{i\theta})}}$ (making heavy use of the fact that $z\overline{z} = |z|^2$); and for any distinct complex numbers $a, b \in \mathbb{C}$ we have

$$\frac{a}{a-b} - \frac{\overline{b}}{\overline{(b-a)}} = \frac{a(\overline{b}-\overline{a})-\overline{b}(a-b)}{(a-b)\overline{(b-a)}} = -\frac{|b|^2-|a|^2}{|b-a|^2} = \frac{|a^2-|b|^2}{|b-a|^2}$$

and

$$\begin{aligned}
\overline{\left(\frac{a+b}{a-b}\right)} = \frac{\overline{(a+b)(\overline{a-b})}}{|a-b|^2} = \frac{\overline{(a+b)}(a-b)}{|a-b|^2} = \frac{\overline{a+b}}{\overline{(a-b)}} \\
\implies 2\,\mathfrak{Re}\left(\frac{a+b}{a-b}\right) = \frac{a+b}{a-b} + \frac{\overline{a+b}}{\overline{(a-b)}} = \frac{(a+b)(\overline{a}-\overline{b})+(\overline{a}+\overline{b})(a-b)}{|a-b|^2} = \frac{2|a|^2+0+0-2|b|^2}{|a-b|^2} \\
\implies \frac{a}{a-b} - \frac{\overline{b}}{\overline{(b-a)}} = \mathfrak{Re}\left(\frac{a+b}{a-b}\right) \\
\implies \frac{re^{i\theta}}{re^{i\theta}-z} - \frac{re^{i\theta}}{re^{i\theta}-z'} = \frac{re^{i\theta}}{re^{i\theta}-z} - \frac{\overline{z}}{\overline{(z-re^{i\theta})}} = \mathfrak{Re}\left(\frac{re^{i\theta}+z}{re^{i\theta}-z}\right).
\end{aligned}$$

Combining everything together we have

$$\begin{aligned}
f(z) - 0 &= \frac{1}{2\pi i}\int_{\gamma_{0,r,\circlearrowleft}} \frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\int_{\gamma_{0,r,\circlearrowleft}} \frac{f(w)}{w-z'}\,dw \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})\cdot\left(\frac{re^{i\theta}}{re^{i\theta}-z} - \frac{re^{i\theta}}{re^{i\theta}-z'}\right)d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})\cdot\mathfrak{Re}\left(\frac{re^{i\theta}+z}{re^{i\theta}-z}\right)d\theta.
\end{aligned}$$

As I said before, taking real parts yields the Poisson integral formula for harmonic $u : U \to \mathbb{R}$.

Exercise S&S §2.1

(Fresnel integrals) We are asked to compute $\int_0^\infty \sin(x^2)\,dx$ and $\int_0^\infty \cos(x^2)\,dx$. S&S permits us to use the identity $\int_{-\infty}^\infty e^{-x^2}\,dx = \sqrt{\pi}$ (hence $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$ since e^{-x^2} is even), and aslo provides a hint to integrate $f(z) := e^{-z^2}$ over the path $[0 \to R]$, along a circle from R to $Re^{i\pi/4}$, and $[Re^{i\pi/4} \to 0]$.

We can parameterize these paths as follows: $\gamma_1 := [t \mapsto t] : [0, R] \to \mathbb{C}; \gamma_2 := [t \mapsto Re^{it}] : [0, \frac{\pi}{4}] \to \mathbb{C}; \gamma_3 := [t \to (R - t)e^{i\pi/4}] : [0, R] \to \mathbb{C}$. Using the "change of variables formula" (Ex. 17(iv) of Notes 1 which we did last week), we have on one hand that

$$\begin{aligned} \oint_{\gamma_1+\gamma_2+\gamma_3} f(z)\, dz &= \int_0^R e^{-t^2}\, dt + \int_0^{\pi/4} \exp(-R^2 e^{2it})(iRe^{it})\, dt + \int_0^R \exp(-(R-t)^2 e^{i\pi/2})(-e^{i\pi/4})\, dt \\ &= \int_0^R e^{-t^2}\, dt + iR \int_0^{\pi/4} \exp(-R^2 e^{2it} + it)\, dt + \int_R^0 \exp(-u^2 i)\big(-\tfrac{1+i}{\sqrt{2}}\big)(-\,du) \end{aligned}$$

(for any $R > 0$); and on the other hand because $f(z) = e^{-z^2}$ is holomorphic on all of $\mathbb{C}$ and $\gamma_1 + \gamma_2 + \gamma_3$ is a closed curve contractible in $\mathbb{C}$, Cauchy's theorem tells us that the LHS integral is 0. Rearranging, multiplying by $\frac{1-i}{\sqrt{2}}$ (the multiplicative inverse of $\frac{1+i}{\sqrt{2}}$), and applying absolute values we have

$$\begin{aligned} \left| \int_0^R \exp(-iu^2)\, du - \tfrac{1-i}{\sqrt{2}} \int_0^R e^{-t^2}\, dt \right| &= \left| iR\tfrac{1-i}{\sqrt{2}} \int_0^{\pi/4} \exp(-R^2 e^{2it} + it)\, dt \right| \\ &\leqslant R \int_0^{\pi/4} \big|\exp(-R^2 e^{2it})\big| \cdot |e^{it}|\, dt \\ &= R \int_0^{\pi/4} \exp(\mathfrak{Re}(-R^2 e^{2it}))\, dt \\ &= R \int_0^{\pi/4} \exp(R^2 \cdot (-\cos(2t)))\, dt \\ &\leqslant R \int_0^{\pi/4} \exp(R^2 \cdot (-(1 - \tfrac{4}{\pi}t)))\, dt \\ &= R \cdot \tfrac{\pi}{4R^2} \exp(\tfrac{4R^2}{\pi}t - R^2)\big|_0^{\pi/4} = \tfrac{\pi}{4R}\big(1 - e^{-R^2}\big) \end{aligned}$$

where we used that $|e^z| = e^{\mathfrak{Re}(z)}$; the increasing nature of e^x; and the inequality $\cos(2x) \geqslant 1 - \frac{4}{\pi}x$ on $[0, \frac{\pi}{4}]$ (note $1 - \frac{4}{\pi}x$ is the line between $(0, 1)$ and $(\frac{\pi}{4}, 0)$), where this inequality is due to the concavity of $\cos(2x)$ on $[0, \frac{\pi}{4}]$ (i.e. non-positivity of the second derivative $-4\cos 2x$ on $(0, \frac{\pi}{4})$). In case anyone wants a proof of that: if $f \in \mathcal{C}([a, b], \mathbb{R})$, and f'' exists and is $\leqslant 0$ on (a, b), then letting L be line between $(a, f(a))$ and $(b, f(b))$, $f \geqslant L$ on $[a, b]$. We may subtract line at the beginning and reduce to the case where $f(a) = f(b) = 0$ (subtracting line does not impact second derivative f''); then line $L = 0$ on $[a, b]$. So now assume f.s.o.c. that $f(x_0) < 0$ for $x_0 \in (a, b)$. MVT says there exist $x_{-1} \in (a, x_0), x_1 \in (x_0, b)$ s.t. $f'(x_{-1}) < 0, f'(x_1) > 0$. MVT again (which only requires that f'' exists at all points of (a, b)) says $x_2 \in (x_{-1}, x_1)$ s.t. $f''(x_2) > 0$; contradiction.

Anyways, since we may take the fact $\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$ as a given, $|\int_0^R e^{-t^2}\, dt - \frac{\sqrt{\pi}}{2}| \to 0$ as $R \to \infty$, so by triangle inequality and the fact that $\frac{\pi}{4R}\big(1 - e^{-R^2}\big)$ also $\to 0$ as $R \to \infty$, we get that $|\int_0^R \exp(-iu^2)\, du - \frac{1-i}{\sqrt{2}}\frac{\sqrt{\pi}}{2}| \to 0$ as $R \to \infty$. Euler's formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ and even/odd-ness of cos/sin give

$$\int_0^\infty \cos(u^2) - i\sin(u^2)\, du = \int_0^\infty e^{-iu^2}\, du = \frac{\sqrt{2\pi}}{4} - i\frac{\sqrt{2\pi}}{4}.$$

Comparing real and imaginary parts yields the desired claim $\int_0^\infty \cos(u^2)\, du = \int_0^\infty \sin(u^2)\, du = \frac{\sqrt{2\pi}}{4}$.

Problem S&S §2.3(a)

Morera's theorem (as it appears in Theorem 33 in Notes 3) says that for any open $U \subseteq \mathbb{C}$ and continuous $f : U \to \mathbb{C}$, if f is conservative for all polygonal paths, i.e. $\oint_\gamma f(z)\,dz = 0$ for all closed polygonal paths $\gamma \subseteq U$, then f is holomorphic on U. In our proof of Exercise 32 from Notes 2 (which we did last week), we in particular proved that f being conservative for all simple closed triangles (a further weakening of condition (iii) from Ex. 32) $\implies$ f is conservative for all polygonal paths (condition (ii) from Ex. 32), so Morera's theorem holds even if we just assume f is conservative for all simple closed triangles.

In this problem, we are asked to show that Morera's theorem continues to hold if we replace "conservative for all simple closed triangles in U" with "conservative for all circles in U", **with the further assumption** that $f : U \to \mathbb{C}$ is smooth in the $\mathbb{R}^2$ sense, i.e. all its x, y partials exist and are continuous.

We proceed guided by the given hint from S&S. Recall from Ex. 23(i) from Notes 1 (which we did in Homework 2) that f is holomorphic on U if and only if $\partial_{\overline{z}} f := \frac{1}{2}(\partial_x f - \frac{1}{i}\partial_y f)$ vanishes on U (also recall $\partial_z f := \frac{1}{2}(\partial_x f + \frac{1}{i}\partial_y f)$); and from Ex. 23(iii) f is Fréchet differentiable at $z_0 \in U$ if and only if $f(z) = f(z_0) + [\partial_z(f)](z_0)(z - z_0) + [\partial_{\overline{z}}(f)](z_0)(\overline{z - z_0}) + \mathfrak{o}(|z - z_0|)$ (where the $\mathfrak{o}$ is w.r.t. the limit as $z \to z_0$; i.e. $\mathfrak{o}(|z - z_0|)$ represents a function ψ s.t. $\frac{\psi(z)}{|z-z_0|} \to 0 \iff \left|\frac{\psi(z)}{|z-z_0|} - 0\right| \to 0$ as $z \to z_0$).

Using this notation and the notation $\gamma_{z_0,r,\circlearrowleft} := [t \mapsto z_0 + re^{it}] : [0, 2\pi] \to \mathbb{C}$ (a $\mathcal{C}^1$ curve, allowing us to use the "change of variables formula" we proved last week in Ex.17(iv) from Notes 2), we have (also using FTC(Id), i.e. Thm. 28 in Notes 2, which tells us that $\oint_{\gamma_{z_0,r,\circlearrowleft}} f(z_0) + [\partial_z(f)](z_0)(z - z_0)\,dz = 0$ since the integrand is a linear/affine function and hence has an antiderivative):

$$\begin{aligned}
0 = \oint_{\gamma_{z_0,r,\circlearrowleft}} f(z)\,dz &= 0 + [\partial_{\overline{z}}(f)](z_0) \cdot \oint_{\gamma_{z_0,r,\circlearrowleft}} (\overline{z - z_0})\,dz + \oint_{\gamma_{z_0,r,\circlearrowleft}} \psi(z)\,dz \\
&= [\partial_{\overline{z}}(f)](z_0) \cdot \int_0^{2\pi} (\overline{re^{it}}) \cdot ire^{it}\,dt + \oint_{\gamma_{z_0,r,\circlearrowleft}} \psi(z)\,dz \\
&= [\partial_{\overline{z}}(f)](z_0) \cdot \int_0^{2\pi} (re^{-it}) \cdot ire^{it}\,dt + \oint_{\gamma_{z_0,r,\circlearrowleft}} \psi(z)\,dz \\
&= [\partial_{\overline{z}}(f)](z_0) \cdot (i2\pi r^2) + \oint_{\gamma_{z_0,r,\circlearrowleft}} \psi(z)\,dz,
\end{aligned}$$

for any $r > 0$ s.t. $\overline{D(z_0, r)} \subseteq U$ (such $r > 0$ exist because U is open). Because $\frac{\psi(z)}{|z-z_0|} \to 0 \iff \left|\frac{\psi(z)}{|z-z_0|} - 0\right| \to 0$ as $z \to z_0$, it is true that for every $\epsilon > 0$, all small enough $r > 0$ will have $\frac{\psi(z)}{|z-z_0|} < \epsilon$ for all $z \in D(z_0, r)$. Thus for all small enough $r > 0$, the above equality and the ML-estimate yield:

$$2\pi\left|[\partial_{\overline{z}}(f)](z_0)\right| = \tfrac{1}{r^2}\left|-[\partial_{\overline{z}}(f)](z_0) \cdot (i2\pi r^2)\right| = \tfrac{1}{r}\left|\oint_{\gamma_{z_0,r,\circlearrowleft}} \frac{\psi(z)}{|z - z_0|}\,dz\right| \leqslant \tfrac{1}{r} \cdot \epsilon \cdot 2\pi r.$$

In other words, we've found that for every $\epsilon > 0$, $\left|[\partial_{\overline{z}}(f)](z_0)\right| \leqslant \epsilon$ and so $[\partial_{\overline{z}}(f)](z_0) = 0$. But $z_0 \in U$ was arbitrarily chosen, so indeed $\partial_{\overline{z}}(f)$ is 0 on all of U, and we are done.

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Daniel Rui - 10/22/21

Exercise 7 (Notes 2)

Let γ_1, γ_2 be curves with the terminal point of γ_1 equal to the initial point of γ_2. We want to show that $|\gamma_1+\gamma_2| = |\gamma_1|+|\gamma_2|$ (and so in particular $\gamma_1+\gamma_2$ is rectifiable if and only if γ_1, γ_2 are both individually rectifiable). Recall the definition of curve and curve addition (see Def. 1 and the paragraphs after Ex. 2 in Notes 2), namely that $\gamma_i : [a_i, b_i] \to \mathbb{C}$ $(i = 1, 2)$ are continuous functions, and that in the situation that $a_2 = b_1$ (otherwise we can replace γ_2 by $\tilde{\gamma}_2 : ([a_2, b_2]-(b_1-a_2)) \to \mathbb{C}$ defined as $\tilde{\gamma}_2(t) := \gamma_2(t+(b_1-a_2))$), we can define the curve $\gamma_1 + \gamma_2 : [a_1, b_2] \to \mathbb{C}$ by $t \in [a_i, b_i] \implies [\gamma_1 + \gamma_2](t) = \gamma_i(t)$ (note only $a_2 = b_1$ is in both $[a_i, b_i]$, $i = 1, 2$, but the sum is well-defined because we required that $\gamma_1(b_1) = \gamma_2(a_1)$).

Recall also the definition of $|\gamma|$, namely it is the supremum over all partitions $a = t_0 < \ldots < t_n = b$ of $[a, b]$ of the sum $\sum_{j=1}^n |\gamma(t_j) - \gamma(t_{j-1})|$. We use the notation $\Lambda(\gamma, P)$ to denote the above "polygonal approximation arclength" sum for a partition P. Observe that any partition P_1 of $[a_1, b_1]$, say $a_1 = t_0 < \ldots < t_n = b_1$, and any partition P_2 of $[a_2, b_2]$, say $a_2 = s_0 < \ldots < s_m = b_2$, join together to form a partition $a_1 = t_0 < \ldots < t_n = b_1 = a_2 = s_0 < \ldots < s_m = b_2$ (denoted $P = P_1 \cup P_2$) of $[a_1, b_2]$, where furthermore (denoting $\Gamma := \gamma_1 + \gamma_2$) we have $\Lambda(\Gamma, P) = \sum_{j=1}^n |\Gamma(t_j) - \Gamma(t_{j-1})| + \sum_{j=1}^m |\Gamma(s_j) - \Gamma(s_{j-1})| = \sum_{j=1}^n |\gamma_1(t_j) - \gamma_1(t_{j-1})| + \sum_{j=1}^m |\gamma_2(s_j) - \gamma_2(s_{j-1})| = \Lambda(\gamma_1, P_1) + \Lambda(\gamma_2, P_2)$.

Thus for ANY arbitrary lesser values $\ell_1 < |\gamma_1|$ and $\ell_2 < |\gamma_2|$, by the definition of supremum we can find some partitions P_1, P_2 (as above) s.t. $\ell_1 + \ell_2 < \Lambda(\gamma_1, P_1) + \Lambda(\gamma_2, P_2) = \Lambda(\gamma_1 + \gamma_2, P_1 \cup P_2) \leqslant |\gamma_1 + \gamma_2|$, so indeed we have the inequality $|\gamma_1| + |\gamma_2| \leqslant |\gamma_1 + \gamma_2|$ (this works in both cases LHS if finite or $+\infty$).

For the other direction, for any partition P of $[a_1, b_2]$, it may or may not contain the shared point $a_2 = b_1$. If it does, say P is the partition $a_1 = t_0 < \ldots < t_K < \ldots < t_n = b_2$ where $t_K = a_2 = b_1$, then we can split P into two partitions P_1, P_2 of $[a_1, b_1]$ and $[a_2, b_2]$, namely $a_1 = t_0 < \ldots < t_K = b_1$ and $a_2 = t_K < \ldots < t_n = b_2$ respectively. Then, as above, we have that $\Lambda(\Gamma, P) = \Lambda(\gamma_1, P_1)+\Lambda(\gamma_2, P_2)$.

If P does not contain the shared point $a_2 = b_1$, we can create a refinement $\hat{P}$ of P that does contain it, i.e. $\hat{P}$ will be the partition $a_1 = t_0 < \ldots < t_K < \ldots < t_n = b_2$ where $t_K = a_2 = b_1$ and $a_1 = t_0 < \ldots < t_{K-1} < t_{K+1} < \ldots < t_n = b_2$ is the partition P. Then as above we can split this partition into $\hat{P}_1, \hat{P}_2$ and get that $\Lambda(\Gamma, \hat{P}) = \Lambda(\gamma_1, \hat{P}_1) + \Lambda(\gamma_2, \hat{P}_2)$. Because $\hat{P}$ is a refinement of P, the triangle inequality gives that $\Lambda(\Gamma, \hat{P}) \geqslant \Lambda(\Gamma, P)$.

Thus for ANY arbitrary lesser value $\ell < |\gamma_1 + \gamma_2|$, by the definition of supremum we can find a partition P of $[a_1, b_2]$ and another partition $\hat{P}$ (either P itself or a refinement containing the shared point $a_2 = b_1$) s.t. $\ell < \Lambda(\Gamma, \hat{P}) \leqslant \Lambda(\Gamma, \hat{P}) = \Lambda(\gamma_1, \hat{P}_1) + \Lambda(\gamma_2, \hat{P}_2) \leqslant |\gamma_1| + |\gamma_2|$, and so indeed we get the inequality $|\gamma_1 + \gamma_2| \leqslant |\gamma_1| + |\gamma_2|$ (this works in both cases LHS if finite or $+\infty$). Thus we have proven that $|\gamma_1 + \gamma_2| = |\gamma_1| + |\gamma_2|$ (and in particular the LHS is $+\infty$ if and only if the RHS is $+\infty$).

Exercise 11

We show that the curve $\gamma : [0,1] \to \mathbb{C}$ given by $\gamma(t) := t(1 + i\sin(\frac{1}{t}))$ for $t \in (0,1]$ and $\gamma(0) := 0$ is continuous but not rectifiable. It suffices to show that for any "bound" $B > 0$, there is some partition P of $[0,1]$ s.t. the Riemann sum $\Sigma(\gamma, P) := \sum_{t_j \in P\setminus\{0\}} |\gamma(t_j) - \gamma(t_{j-1})|$ is $> B$. Defining $\alpha_n := (2\pi n + \frac{\pi}{2})^{-1}$ and $\beta_n := (2\pi n + \frac{3\pi}{2})^{-1}$ (so $\gamma(\alpha_n) = \alpha_n(1+i)$ and $\gamma(\beta_n) = \beta_n(1-i)$), we can define for any $N \in \mathbb{N}$ a partition $0 < \beta_N < \alpha_N < \ldots < \beta_1 < \alpha_1 < 1$ (which we shall henceforth denote as P_N). Then, (using the fact that $\alpha_n > \beta_n = (2\pi n + \frac{3\pi}{2})^{-1} > (2\pi n + 2\pi)^{-1} = \frac{1}{2\pi(n+1)}$):

$$\begin{aligned}\Sigma(\gamma, P_N) &= |\gamma(\beta_N) - \gamma(0)| + \sum_{n=1}^{N} |\gamma(\alpha_n) - \gamma(\beta_n)| + \sum_{n=1}^{N-1} |\gamma(\beta_n) - \gamma(\alpha_{n+1})| \\ &= 0 + \sum_{n=1}^{N} \big|\mathtt{Im}\,(\gamma(\alpha_n) - \gamma(\beta_n))\big| + 0 \geqslant \sum_{n=1}^{N} \big|\mathtt{Im}\,(\alpha_n(1+i) - \beta_n(1-i))\big| \\ &= \sum_{n=1}^{N} |\alpha_n + \beta_n| \geqslant 2\sum_{n=1}^{N} \tfrac{1}{2\pi(n+1)} = \tfrac{1}{\pi}\sum_{n=2}^{N+1} \tfrac{1}{n}.\end{aligned}$$

But because the sum $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the tail $\sum_{n=2}^{N+1} \frac{1}{n}$ of this sum will grow arbitrarily large as $N \to \infty$, meaning for our fixed (large) "bound" $B > 0$ above, we can always find $N \in \mathbb{N}$ s.t. $\Sigma(\gamma, P_N) \geqslant \frac{1}{\pi}\sum_{n=2}^{N+1} \frac{1}{n} > B$. Thus, the supremum of these sums over all partitions of $[0,1]$ is $+\infty$, and the curve γ is not rectifiable.

Exercise 16

We give an example of a curve (i.e. a continuous function) $\gamma : [0,1] \to \mathbb{C}$ where the Riemann sums $\sum_{j=1}^{n} f(\gamma(t_j^*))[\gamma(t_j) - \gamma(t_{j-1})]$ do NOT converge, even for the almost simplest possible function $f(z) = z$. Recall (see Prop. 15 in Notes 2) that by saying that "the Riemann sums converge" we mean that there is some limit L s.t. for every $\epsilon > 0$ there is $\delta > 0$ s.t. for ANY partition $a = t_0 < \ldots < t_n = b$ of maximal mesh size $\max_{j\in[n]} |t_j - t_{j-1}| \leqslant \delta$ and ANY choice of $t_j^* \in [t_{j-1}, t_j]$ ($j \in \{1, \ldots, n\}$), the Riemann sum is within ϵ of L: $\big|L - \sum_{j=1}^{n} f(\gamma(t_j^*))[\gamma(t_j) - \gamma(t_{j-1})]\big| < \epsilon$.

As is suggested in the exercise statement, we consider a variant of the curve from Exercise 11 above that oscillates even *more* wildly, namely $\gamma : [0,1] \to \mathbb{C}$ defined by $\gamma(t) = \sqrt{t}(1 + i\sin(\frac{1}{t}))$ for $t \in (0,1]$ and $\gamma(0) := 0$. This function is continuous for precisely the same reasons as the curve from Exercise 11 above (more explicitly, continuous on $(0,1]$ because there it's just a sum/product/composition of continuous functions, and continuous at 0 because $|\gamma(t) - 0| \leqslant \sqrt{t}|1 + i\sin(\frac{1}{t})| \leqslant \sqrt{t}(|1| + |\sin(\frac{1}{t})|) \leqslant 2\sqrt{t}$ which goes to 0 as $t \to 0$).

Outline: to prove that the Riemann sums for this curve and the function $f(z) = z$ do not converge, it is more than enough to show that for any $\delta > 0$ and "bound" $B > 0$, we can find a partition $a = t_0 < \ldots < t_n = b$ of maximal mesh size $\max_{j\in[n]} |t_j - t_{j-1}| \leqslant \delta$ and intermediate points $t_j^* \in [t_{j-1}, t_j]$ s.t. the Riemann sum $\sum_{j=1}^{n} \gamma(t_j^*)[\gamma(t_j) - \gamma(t_{j-1})]$ has real part $> B$, and another choice of intermediate points $t_j^* \in [t_{j-1}, t_j]$ w.r.t. the same partition having Riemann sum with real part $< -B$.

First, we fix (small) $\delta > 0$ and (large) $B > 0$. As $(2\pi N + \frac{\pi}{2})^{-1} \to 0$ as $N \to \infty$, there is some $N \in \mathbb{N}$ s.t. $n \geqslant N \implies (2\pi n + \frac{\pi}{2})^{-1} < \delta$. Let us denote by p_N the "pivot" $p_N := (2\pi N + \frac{\pi}{2})^{-1}$ (I'll explain the choice of word in a moment). Because γ restricted to $[p_N, 1]$ is a $\mathcal{C}^1$ curve (in fact it's $\mathcal{C}^1$ on $(0, 1]$ since on that interval it's just a composition of $\mathcal{C}^1$ functions), so by Prop. 10 in Notes 2 the restriction $\gamma^1_{p_N} := \gamma|_{[p_N,1]}$ is a rectifiable curve, and Prop. 15 in Notes 2 gives that the Riemann sums of $f(z) = z$ on $\gamma^1_{p_N}$ DO converge to some limit $L^1_{p_N} \in \mathbb{C}$, otherwise known/denoted as $L^1_{p_N} = \int_{\gamma^1_{p_N}} z\,dz$.

Thus there is some partition P_R ("partition right of the pivot") of $[p_N, 1]$ of maximum mesh size $< \delta$ s.t. the Riemann sum $\Sigma[P_R] := \sum_{t_j \in P_R \setminus \{p_N\}} \gamma(t_j)[\gamma(t_j) - \gamma(t_{j-1})]$ (so t_j runs over all times in P_R except the first time/left endpoint p_N) is within ϵ of $L^1_{p_N}$ (in absolute value), for some small $\epsilon > 0$ we fix beforehand.

Partition Left of Pivot

Now we discuss a "partition left of the pivot", P_L. Using the notation $\alpha_n := (2\pi n + \frac{\pi}{2})^{-1}$, we can define, for any integer $M > N$, a partition ${}_\alpha P^M_L$ of $[0, p_N]$ by $0 < \alpha_M < \ldots < \alpha_{N+1} < \alpha_N = p_N$. Note that $\gamma(\alpha_n) = \sqrt{\alpha_n}(1 + i)$. Using the notation $\beta_n := (2\pi n + \frac{3\pi}{2})^{-1}$ (so $\gamma(\beta_n) = \sqrt{\beta_n}(1 - i)$), we have that $\beta_M < \ldots < \beta_{N+1} < \beta_N$, so by interlacing this with ${}_\alpha P^M_L$ we can also define (for the $M > N$ chosen above) the partition P^M_L of $[0, p_N]$ by $0 =: \alpha_\infty < \beta_M < \alpha_M < \ldots < \alpha_{N+1} < \beta_N < \alpha_N = p_N$. Because $p_N < \delta$, the maximum mesh size of this partition is definitely $< \delta$. The Riemann sum $\Sigma_L[P^M_L] := \sum_{t_j \in P^M_L \setminus \{0\}} \gamma(t_{j-1})[\gamma(t_j) - \gamma(t_{j-1})]$ w.r.t. this partition (i.e. choosing t^*_j to always be LEFT endpoint of $[t_{j-1}, t_j]$) is

$$\begin{aligned}
\Sigma_L\left[P^M_L\right] &= \gamma(0)\left[\gamma(\beta_M) - \gamma(0)\right] + \sum_{n=N}^{M} \gamma(\beta_n)\left[\gamma(\alpha_n) - \gamma(\beta_n)\right] + \sum_{n=N+1}^{M} \gamma(\alpha_n)\left[\gamma(\beta_{n-1}) - \gamma(\alpha_n)\right] \\
&= 0 + \sum_{n=N}^{M} \sqrt{\beta_n}(1-i)\left[\sqrt{\alpha_n}(1+i) - \sqrt{\beta_n}(1-i)\right] + \sum_{n=N+1}^{M} \sqrt{\alpha_n}(1+i)\left[\sqrt{\beta_{n-1}}(1-i) - \sqrt{\alpha_n}(1+i)\right] \\
&= \sum_{n=N}^{M} \left[\sqrt{\beta_n\alpha_n}(2) - \beta_n(0-2i)\right] + \sum_{n=N+1}^{M} \left[\sqrt{\alpha_n\beta_{n-1}}(2) - \alpha_n(0+2i)\right]
\end{aligned}$$

Then, we have (using the fact that $\alpha_n > \beta_n = (2\pi n + \frac{3\pi}{2})^{-1} > (2\pi n + 2\pi)^{-1} = \frac{1}{2\pi(n+1)}$):

$$\begin{aligned}
\mathfrak{Re}(\Sigma_L(P^M_L)) &= 2\left(\sum_{n=N}^{M} \sqrt{\beta_n\alpha_n} + \sum_{n=N+1}^{M} \sqrt{\alpha_n\beta_{n-1}}\right) \\
&\geqslant 2\left(\sum_{n=N}^{M} \sqrt{\frac{1}{2\pi(n+1)} \cdot \frac{1}{2\pi(n+1)}} + \sum_{n=N+1}^{M} \sqrt{\frac{1}{2\pi(n+1)} \cdot \frac{1}{2\pi n}}\right) \\
&\geqslant 2\left(\sum_{n=N+1}^{M} \frac{1}{2\pi(n+1)} + \sum_{n=N+1}^{M} \frac{1}{2\pi(n+1)}\right) = \frac{2\cdot 2}{2\pi} \sum_{n=N+1}^{M} \frac{1}{n+1} = \frac{2}{\pi} \sum_{n=N+2}^{M+1} \frac{1}{n}.
\end{aligned}$$

But because the sum $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, any tail $\sum_{n=N}^{M} \frac{1}{n}$ of this sum will grow arbitrarily large as $M \to \infty$. Thus, fixing any (large) $\tilde{B} > 0$, by taking any integer $M > N$ very large above, we will get that $\mathfrak{Re}(\Sigma_L(P^M_L)) > \tilde{B}$ (we will relate this $\tilde{B}$ to the "bound" B that we fixed at the beginning in a moment).

On the other hand, taking the Riemann sum $\Sigma_R[P_L^M]$ w.r.t. this partition with t_j^* chosen to always be RIGHT endpoint of $[t_{j-1}, t_j]$, except the first $t_1^* := 0$), we have

$$\begin{aligned}\Sigma_R[P_L^M] &= \gamma(0)[\gamma(\beta_M) - \gamma(0)] + \sum_{n=N}^{M} \gamma(\alpha_n)[\gamma(\alpha_n) - \gamma(\beta_n)] + \sum_{n=N+1}^{M} \gamma(\beta_{n-1})[\gamma(\beta_{n-1}) - \gamma(\alpha_n)] \\ &= \sum_{n=N}^{M} \sqrt{\alpha_n}(1+i)[\sqrt{\alpha_n}(1+i) - \sqrt{\beta_n}(1-i)] + \sum_{n=N+1}^{M} \sqrt{\beta_{n-1}}(1-i)[\sqrt{\beta_{n-1}}(1-i) - \sqrt{\alpha_n}(1+i)] \\ &= \sum_{n=N}^{M} [\alpha_n(0+2i) - \sqrt{\alpha_n\beta_n}(2)] + \sum_{n=N+1}^{M} [\beta_{n-1}(0-2i) - \sqrt{\beta_{n-1}\alpha_n}(2)]\end{aligned}$$

and so (the RHS is the same inequality as above)

$$-\mathfrak{Re}(\Sigma_R[P_L^M]) = 2\left(\sum_{n=N}^{M} \sqrt{\beta_n\alpha_n} + \sum_{n=N+1}^{M} \sqrt{\alpha_n\beta_{n-1}}\right) \geqslant \frac{2}{\pi} \sum_{n=N+2}^{M+1} \frac{1}{n},$$

so indeed for all very large $M > N$ (like above), $\mathfrak{Re}(\Sigma_R[P_L^M]) < -\tilde{B}$.

Finale

Thus finally, for any integer $M > N$ we can define the partition $P^M := P_L^M \cup P_R$, stitching together the two partitions above since they both share the "pivot" point p_N. Then we can combine the two pairs of Riemann sums: $\Sigma_L[P_L^M] \,\&\, \Sigma[P_R]$, and $\Sigma_R[P_L^M] \,\&\, \Sigma[P_R]$ into two Riemann sums of the partition P^M (of maximal mesh size $\leqslant \delta$), which we shall denote as $\Sigma^+[P^M] := \Sigma_L[P_L^M] + \Sigma[P_R]$ and $\Sigma^-[P^M] := \Sigma_R[P_L^M] + \Sigma[P_R]$.

Now recall from the above discussion of the "partition right of the pivot" P_R that we have some complex number $L_{p_N}^1$ s.t. $|\mathfrak{Re}(\Sigma[P_R]) - \mathfrak{Re}(L_{p_N}^1)| \leqslant |\Sigma[P_R] - L_{p_N}^1| < \epsilon$ (first inequality by triangle inequality, or Pythagorean theorem), and recall from the above subsection "Partition Left of Pivot" that for any (large) $\tilde{B} > 0$, we have some (large) integer $M > N$ s.t. $\mathfrak{Re}(\Sigma_L[P_L^M]) > \tilde{B}$ and $\mathfrak{Re}(\Sigma_R[P_L^M]) < -\tilde{B}$. Thus, we have

$$\mathfrak{Re}(\Sigma^+[P^M]) = \mathfrak{Re}(\Sigma_L[P_L^M]) + \mathfrak{Re}(\Sigma[P_R]) > \tilde{B} + \mathfrak{Re}(L_{p_N}^1) - \epsilon,$$

and similarly

$$\mathfrak{Re}(\Sigma^-[P^M]) = \mathfrak{Re}(\Sigma_R[P_L^M]) + \mathfrak{Re}(\Sigma[P_R]) < -\tilde{B} + \mathfrak{Re}(L_{p_N}^1) + \epsilon.$$

As $\mathfrak{Re}(L_{p_N}^1)$ and ϵ are just constants, we can pick $\tilde{B}$ beforehand large enough s.t. for the "bound" $B > 0$ we fixed at the very beginning, we have

$$\mathfrak{Re}(\Sigma^+[P^M]) > \tilde{B} + \mathfrak{Re}(L_{p_N}^1) - \epsilon > B \quad \text{and} \quad \mathfrak{Re}(\Sigma^-[P^M]) < -\tilde{B} + \mathfrak{Re}(L_{p_N}^1) + \epsilon < -B.$$

We have proven what we promised to prove in the **Outline**, so we are done.

Exercise 17(iv)

If $\gamma : [a, b] \to \mathbb{C}$ is $\mathcal{C}^1$ (i.e. continuously differentiable) and $f : \gamma([a, b]) \to \mathbb{C}$ is some continuous function, then we have a formula for the contour integral:

$$\int_\gamma f(z)\, dz = \int_a^b f(\gamma(t))\gamma'(t)\, dt.$$

We prove this as follows: it suffices to show that the Riemann sums $\sum_{j=1}^n f(\gamma(t_j^*))[\gamma(t_j) - \gamma(t_{j-1})]$ converge to $\int_a^b f(\gamma(t))\gamma'(t)\, dt$, since by uniqueness of limits, $L := \int_\gamma f(z)\, dz$ (which we defined to be the limiting value of such Riemann sums) must equal $\int_a^b f(\gamma(t))\gamma'(t)\, dt$. Recall (see Prop. 15 in Notes 2 or Exercise 16 above) that by saying that "the Riemann sums converge to L" we mean that for every $\epsilon > 0$ there is $\delta > 0$ s.t. for ANY partition $a = t_0 < \ldots < t_n = b$ of maximal mesh size $\max_{j\in[n]} |t_j - t_{j-1}| \leqslant \delta$ and ANY choice of $t_j^* \in [t_{j-1}, t_j]$ $(j \in \{1, \ldots, n\})$, the Riemann sum is within ϵ of L: $\left|L - \sum_{j=1}^n f(\gamma(t_j^*))[\gamma(t_j) - \gamma(t_{j-1})]\right| < \epsilon$.

First, note that since $\gamma(t) =: x(t) + iy(t)$ is a $\mathcal{C}^1$ complex-valued function on $[a, b]$, the component functions $x, y : [a, b] \to \mathbb{R}$ must also be $\mathcal{C}^1$ ($x(t), y(t)$ are differentiable and satisfy $\gamma'(t) = x'(t) + iy'(t)$ because the difference quotient of γ decomposes into difference quotients in each independent component, in this case the real and imaginary parts $x(t)$ and $y(t)$, where the limit of this difference quotient (i.e. the derivative) for γ exists iff the limit in each independent component exists; and $x'(t), y'(t)$ are continuous be the same argument "a limit of for vector valued function exists iff that limit in each independent component exists"). In particular since $[a, b]$ is a compact set, $x', y', \gamma', \gamma, f$ are all uniformly continuous on $[a, b]$ and there is $M > 0$ s.t. $|f(\gamma(t))|, |\gamma'(t)| \leqslant M$ on $[a, b]$, so for any fixed $\epsilon > 0$, there is $\eta > 0$ s.t. $|z - w| < \eta \implies |f(z) - f(w)| < \frac{\epsilon}{4M(b-a)}$, and $\delta > 0$ s.t. $|t - s| < \delta \implies |x'(t) - x'(s)|, |y'(t) - y'(s)|, |\gamma'(t) - \gamma'(s)|, |\gamma(t) - \gamma(s)| < \min\{\eta, \frac{\epsilon}{4M(b-a)}\}$.

Observe now that $\frac{\gamma(t_j)-\gamma(t_{j-1})}{t_j-t_{j-1}} = \frac{x(t_j)-x(t_{j-1})}{t_j-t_{j-1}} + i\big(\frac{y(t_j)-y(t_{j-1})}{t_j-t_{j-1}}\big)$, so because $x(t), y(t)$ are differentiable real valued functions, the mean value theorem tells us that there are ${}_xt_j^*, {}_yt_j^* \in (t_{j-1}, t_j)$ s.t. the RHS becomes $x'({}_xt_j^*) + iy'({}_yt_j^*)$. Then, we have the following bound on the "error", assuming that $|t_j - t_{j-1}| < \delta$ (defined above in red):

$$\begin{aligned}
\varepsilon(t_{j-1}, t_j) &:= \left|f(\gamma(t_j^*))[\gamma(t_j) - \gamma(t_{j-1})] - \int_{t_{j-1}}^{t_j} f(\gamma(t))\gamma'(t)\, dt\right| \\
&= \left|f(\gamma(t_j^*))[\gamma(t_j) - \gamma(t_{j-1})] - \int_{t_{j-1}}^{t_j} f(\gamma(t_j^*))\gamma'(t_j)\, dt + \int_{t_{j-1}}^{t_j} f(\gamma(t_j^*))\gamma'(t_j)\, dt - \int_{t_{j-1}}^{t_j} f(\gamma(t))\gamma'(t)\, dt\right| \\
&= |f(\gamma(t_j^*))| \cdot (t_j - t_{j-1}) \cdot \left|\frac{\gamma(t_j) - \gamma(t_{j-1})}{t_j - t_{j-1}} - \gamma'(t_j)\right| + \int_{t_{j-1}}^{t_j} \left|f(\gamma(t_j^*))\gamma'(t_j) - f(\gamma(t))\gamma'(t)\right| dt.
\end{aligned}$$

The first term T_1 of the sum on the RHS can be bounded (using that $|t_j - {}_xt_j^*|, |t_j - {}_yt_j^*| \leqslant |t_j - t_{j-1}| < \delta$):

$$\begin{aligned}
T_1 &= |f(\gamma(t_j^*))| \cdot (t_j - t_{j-1}) \cdot \left|x'({}_xt_j^*) + iy'({}_yt_j^*) - (x'(t_j) + iy'(t_j))\right| \\
&= M \cdot (t_j - t_{j-1}) \cdot \left(\left|x'({}_xt_j^*) - x'(t_j)\right| + \left|y'({}_yt_j^*) - y'(t_j))\right|\right) \leqslant M \cdot (t_j - t_{j-1}) \cdot 2 \cdot \tfrac{\epsilon}{4M(b-a)}.
\end{aligned}$$

The second term T_2 of the RHS of the bound for the "error" $\varepsilon(t_{j-1}, t_j)$ can be also bounded (again using properties of δ, η, etc. defined above):

$$\begin{aligned} T_2 &\leqslant \int_{t_{j-1}}^{t_j} \left|f(\gamma(t_j^*))\gamma'(t_j) - f(\gamma(t))\gamma'(t_j)\right| + \left|f(\gamma(t))\gamma'(t_j) - f(\gamma(t))\gamma'(t)\right| dt \\ &\leqslant |\gamma'(t_j)| \cdot \int_{t_{j-1}}^{t_j} \left|f(\gamma(t_j^*)) - f(\gamma(t))\right| dt + \int_{t_{j-1}}^{t_j} |f(\gamma(t))| \cdot |\gamma'(t_j) - \gamma'(t)| \, dt \\ &\leqslant M \cdot \int_{t_{j-1}}^{t_j} \tfrac{\epsilon}{4M(b-a)} \, dt + \int_{t_{j-1}}^{t_j} M \cdot \tfrac{\epsilon}{4M(b-a)} \, dt = \frac{\epsilon}{2(b-a)}(t_j - t_{j-1}). \end{aligned}$$

Adding up the bounds on T_1, T_2 we get

$$\varepsilon(t_{j-1}, t_j) \leqslant T_1 + T_2 = \frac{\epsilon}{(b-a)}(t_j - t_{j-1}).$$

Since the only fact about t_{j-1}, t_j we used was that $|t_j - t_{j-1}| < \delta$, we have that for ANY partition $a = t_0 < \ldots < t_n = b$ of maximal mesh size $\max_{j\in[n]} |t_j - t_{j-1}| < \delta$ and ANY choice of $t_j^* \in [t_{j-1}, t_j]$, the difference between the Riemann sum and the desired integral formula is

$$\left| \sum_{j=1}^{n} f(\gamma(t_j^*))[\gamma(t_j) - \gamma(t_{j-1})] - \int_a^b f(\gamma(t))\gamma'(t) \, dt \right| \leqslant \sum_{j=1}^{n} \varepsilon(t_{j-1}, t_j) \leqslant \sum_{j=1}^{n} \frac{\epsilon}{(b-a)}(t_j - t_{j-1}) = \epsilon,$$

and so we are done.

Exercise 17(ix)

Let U be an open neighborhood of $\gamma([a,b])$, let $\phi : U \to \mathbb{C}$ be a holomorphic function (we are allowed to assume $\mathcal{C}^1$, but we will try to stick mainly with just differentiability), and let $g : \phi(\gamma([a,b])) \to \mathbb{C}$ be a continuous function. We first claim that $\phi \circ \gamma$ is rectifiable, and then once we know that, we can define a contour integral on that curve which turns out to satisfy the formula

$$\int_{\phi\circ\gamma} g(w) \, dw = \int_\gamma g(\phi(z))\phi'(z) \, dz.$$

First, rectifiability of $\phi \circ \gamma$. Given any partition $a = t_0 < \ldots < t_n = b$ of $[a,b]$ (abbreviated by P), we can form the "polygonal approximation arclength" sum $\Lambda(\phi \circ \gamma, P) := \sum_{j=1}^n |\phi(\gamma(t_j)) - \phi(\gamma(t_{j-1}))|$. We would like to bound this perhaps by some constant C times $\Lambda(\gamma, P) := \sum_{j=1}^n |\gamma(t_j) - \gamma(t_{j-1})|$, since we know the sup of those sums (i.e. precisely the arclength $|\gamma|$) is finite, and so then the sup of the sums $\Lambda(\phi \circ \gamma, P)$ would also have to be finite (in fact $\leqslant C|\gamma|$), since the partition P was arbitrarily chosen. This is essentially asking that ϕ be C-Lipschitz.

It suffices to do what is outlined in the previous paragraph for all partitions P of maximum mesh size $\max_{j\in[n]} |t_j - t_{j-1}| < \delta$ for any fixed $\delta > 0$, since by the triangle inequality, $\Lambda(\bullet, \hat{P}) \geqslant \Lambda(\bullet, P)$ for any refinement $\hat{P}$ of P (and we can always find a refinement $\hat{P}$ of P with maximum mesh size $< \delta$), and we really only care about "the largest sums" Λ because arclength is defined as the supremum of such Λ sums.

Since γ is a continuous function on a compact set $[a, b]$, it is uniformly continuous and its image $\operatorname{im}(\gamma) \subseteq U \subseteq \mathbb{C}$ is compact. Because $\operatorname{im}(\gamma)$ is compact and $U^{\complement}$ is closed and disjoint compact and closed sets are distant, then defining $r := \frac{1}{2}d(\operatorname{im}(\gamma), U^{\complement})$, we have that $B(\gamma(t), r) \subseteq U$ for all $t \in [a, b]$, and moreover there is a compact set $K \subseteq U$ that contains all such balls $B(\gamma(t), r)$, namely $K := \{x \in \mathbb{C} : d(x, \operatorname{im}(\gamma)) \leqslant \frac{3}{2}r\}$.

To elaborate on why K has these properties, it is not hard to show that $d(x, \operatorname{im}(\gamma))$ is continuous in x, so K is closed because it's the inverse image of a closed set $[0, \frac{3}{2}r]$ under a continuous function; K is bounded because $\operatorname{im}(\gamma)$ bounded in say $B(0, R)$ implies that $K \subseteq B(0, R + 2r)$; $K \subseteq U \iff K \cap U^{\complement} = \varnothing$ because all elements of $U^{\complement}$ have distance $\geqslant 2r$ from $\operatorname{im}(\gamma)$; and finally it is clear from the definition that $B(\gamma(t), r)) \subseteq K$ for all $t \in [a, b]$.

Because γ is uniformly continuous on $[a, b]$ (continuous on compact set), there is some $\delta > 0$ s.t. $|t - s| < \delta \implies |\gamma(t) - \gamma(s)| < r$. Then for ANY partition P of maximum mesh size $< \delta$ (or we could take $\leqslant \frac{1}{2}\delta$), $|\gamma(t_j) - \gamma(t_{j-1})| < r$, so the straight line segment $[\gamma(t_{j-1}) \to \gamma(t_j)]$ is contained in $B(\gamma(t_j), r)$ by convexity of the ball. Thus, all the straight line segments $[\gamma(t_{j-1}) \to \gamma(t_j)]$, $j \in [n]$, are contained in $\bigcup_{t\in[a,b]} B(\gamma(t), r) \subseteq K$.

Now recall the mean value theorem on $\mathbb{R}^n$: for a Fréchet differentiable function $f : U \to \mathbb{R}$ ($U \subseteq \mathbb{R}^n$ open) and points $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ s.t. the line segment $[\mathbf{a} \to \mathbf{b}]$ between them is contained in U, then there is some ξ on the line segment $[\mathbf{a} \to \mathbf{b}]$ s.t. $[\nabla f](\xi) \cdot (\mathbf{b} - \mathbf{a}) = f(\mathbf{b}) - f(\mathbf{a})$, and in particular by Cauchy-Schwarz, $|f(\mathbf{b}) - f(\mathbf{a})| \leqslant |[\nabla f](\xi)| \cdot |\mathbf{b} - \mathbf{a}|$. Applying this to $u := \mathfrak{Re}(\phi), v := \mathfrak{Im}(\phi)$ (which are indeed Fréchet differentiable functions we can view as being from $\mathbb{R}^2 \to \mathbb{R}^2$, where $\phi'(z) = [\partial_x \phi](z) = [\partial_x u](z) + i[\partial_x v](z)$ and so using the Cauchy-Riemann equations $\partial_x u = \partial_y v$ and $\partial_y u = -\partial_x v$, we get $[\nabla u](z) = \left[\begin{smallmatrix} \mathfrak{Re}(\phi'(z)) \\ -\mathfrak{Im}(\phi'(z)) \end{smallmatrix}\right]$ and $[\nabla v](z) = \left[\begin{smallmatrix} \mathfrak{Im}(\phi'(z)) \\ \mathfrak{Re}(\phi'(z)) \end{smallmatrix}\right]$) yields

$$\begin{aligned}
|\phi(\gamma(t_j)) - \phi(\gamma(t_{j-1}))| &\leqslant |u(\gamma(t_j)) - u(\gamma(t_{j-1}))| + |v(\gamma(t_j)) - v(\gamma(t_{j-1}))| \\
&\leqslant \Big(\big|[\nabla u](\xi_1)\big| + \big|[\nabla v](\xi_2)\big|\Big) \cdot |\gamma(t_{j-1}) - \gamma(t_{j-1})| \\
&\leqslant \Big(\sqrt{|\phi'(\xi_1)|^2 + |\phi'(\xi_1)|^2} + \sqrt{|\phi'(\xi_2)|^2 + |\phi'(\xi_2)|^2}\Big) \cdot |\gamma(t_{j-1}) - \gamma(t_{j-1})| \\
&= \sqrt{2} \cdot \Big(|\phi'(\xi_1)| + |\phi'(\xi_2)|\Big) \cdot |\gamma(t_{j-1}) - \gamma(t_{j-1})|,
\end{aligned}$$

where ξ_1, ξ_1 are on the line segment $[\gamma(t_{j-1}) \to \gamma(t_j)] \subseteq K$. But K is compact, and so the continuous function $|\phi'|$ is bounded by say B on K! So we have proven that for any partition P of maximal mesh size $< \delta$, $\Lambda(\phi \circ \gamma, P) \leqslant \sqrt{2}B\Lambda(\gamma, P) \leqslant \sqrt{2}B|\gamma|$, and so indeed $\phi \circ \gamma$ is rectifiable.

Change of Variables Formula

Prop. 15 in Notes 2 tells us that for rectifiable curves γ and continuous functions $f : \gamma([a, b]) \to \mathbb{C}$, the Riemann sums converge, in the sense that there is some limit $L =: \int_\gamma f(z)\,dz$ s.t. for every $\epsilon > 0$ there is $\delta > 0$ s.t. for ANY partition $a = t_0 < \ldots < t_n = b$ of maximal mesh size $< \delta$ and ANY choice of $t_j^* \in [t_{j-1}, t_j]$, the Riemann sum is within ϵ of L: $\big|L - \sum_{j=1}^n f(\gamma(t_j^*))[\gamma(t_j) - \gamma(t_{j-1})]\big| < \epsilon$.

Thus, to prove the desired integral equality, we need only consider Riemann sum approximations of those integral where the "t_j^*" are taken to be "t_j", since the above-quoted Prop. 15 says that any choice $t_j^* \in [t_{j-1}, t_j]$ leads to convergence. Let us use the notation $\Sigma(f, \gamma, \{t_j\}_{j=1}^n) := \sum_{j=1}^n f(\gamma(t_j))[\gamma(t_j) - \gamma(t_{j-1})]$. Our first step is to disembowel the expression $[\phi(\gamma(t_j)) - \phi(\gamma(t_{j-1}))]$. Using the notation from the above paragraph on the $\mathbb{R}^n$ mean value theorem, we have this expression equals $[u(\gamma(t_j)) - u(\gamma(t_{j-1}))] + i[v(\gamma(t_j)) - v(\gamma(t_{j-1}))]$, and so applying the $\mathbb{R}^2$ mean value theorem here we get

$$[u(\gamma(t_j)) - u(\gamma(t_{j-1}))] = \mathfrak{Re}(\phi'(\xi_1)) \cdot \mathfrak{Re}(\gamma(t_j) - \gamma(t_{j-1})) - \mathfrak{Im}(\phi'(\xi_1)) \cdot \mathfrak{Im}(\gamma(t_j) - \gamma(t_{j-1}))$$

and

$$[v(\gamma(t_j)) - v(\gamma(t_{j-1}))] = \mathfrak{Im}(\phi'(\xi_2)) \cdot \mathfrak{Re}(\gamma(t_j) - \gamma(t_{j-1})) + \mathfrak{Re}(\phi'(\xi_2)) \cdot \mathfrak{Im}(\gamma(t_j) - \gamma(t_{j-1}))$$

where again ξ_1, ξ_2 are on the line segment $[\gamma(t_{j-1}) \to \gamma(t_j)]$. Observe that

$$\begin{aligned}\phi'(\gamma(t_j))[\gamma(t_j) - \gamma(t_{j-1})] = {} & \mathfrak{Re}(\phi'(\gamma(t_j))) \cdot \mathfrak{Re}(\gamma(t_j) - \gamma(t_{j-1})) - \mathfrak{Im}(\phi'(\gamma(t_j))) \cdot \mathfrak{Im}(\gamma(t_j) - \gamma(t_{j-1})) \\ & + i\,\mathfrak{Im}(\phi'(\gamma(t_j))) \cdot \mathfrak{Re}(\gamma(t_j) - \gamma(t_{j-1})) + i\,\mathfrak{Re}(\phi'(\gamma(t_j))) \cdot \mathfrak{Im}(\gamma(t_j) - \gamma(t_{j-1})),\end{aligned}$$

and so the difference $\varepsilon'(t_{j-1}, t_j) := |[\phi(\gamma(t_j)) - \phi(\gamma(t_{j-1}))] - \phi'(\gamma(t_j))[\gamma(t_j) - \gamma(t_{j-1})]|$ is bounded by

$$\varepsilon'(t_{j-1}, t_j) \leqslant \Big(2|\phi'(\xi_1) - \phi'(t_j)| + 2|\phi'(\xi_2) - \phi'(t_j)|\Big) \cdot |\gamma(t_j) - \gamma(t_{j-1})|.$$

Now for our horde of quantifiers: let B denote some large "bound" that we'll specify later. For any $\epsilon > 0$, by uniform continuity of ϕ' on $\text{im}(\gamma)$ (we are assuming ϕ' is continuous on compact $\text{im}(\gamma)$) there is some $\eta > 0$ s.t. $|z - w| < \eta \implies |\phi'(z) - \phi'(w)| < \frac{\epsilon}{B}$. By uniform continuity of γ on $[a, b]$, there is $\delta > 0$ s.t. $|t_j - t_{j-1}| < \delta \implies |\gamma(t_j) - \xi_i| \leqslant |\gamma(t_j) - \gamma(t_{j-1})| < \eta$. For ANY partition $P = \{a = t_0 < \ldots < t_n = b\}$ with maximal mesh size $< \delta$, we therefore get $\varepsilon'(t_{j-1}, t_j) \leqslant 4 \cdot \frac{\epsilon}{B} \cdot |\gamma(t_j) - \gamma(t_{j-1})|$. Let us now define

$$\begin{aligned}\varepsilon(t_{j-1}, t_j) := {} & \Big| g(\phi(\gamma(t_j))) \cdot [\phi(\gamma(t_j)) - \phi(\gamma(t_{j-1}))] - g(\phi(\gamma(t_j))) \cdot \phi'(\gamma(t_j)) \cdot [\gamma(t_j) - \gamma(t_{j-1})] \Big| \\ = {} & |g(\phi(\gamma(t_j)))| \cdot \varepsilon'(t_{j-1}, t_j) \leqslant M \cdot \frac{4\epsilon}{B} \cdot |\gamma(t_j) - \gamma(t_{j-1})|,\end{aligned}$$

where M is s.t. the continuous function $|g| \leqslant M$ on the compact set $\phi(\gamma([a, b]))$. Finally, by Prop. 15 in Notes 2 (discussed at the beginning of this subsection) there is $\tilde{\delta} \in (0, \delta)$ s.t. $P = \{a = t_0 < \ldots < t_n = b\}$ with maximal mesh size $< \tilde{\delta} < \delta$, all Riemann sums $\Sigma(g, \phi \circ \gamma, \{t_j\}_{j=1}^n)$ and $\Sigma((g \circ \phi) \cdot \phi', \gamma, \{t_j\}_{j=1}^n)$ are within $\frac{\epsilon}{3}$ of their respective integrals $\int_{\phi \circ \gamma} g(w)\, dw$ and $\int_\gamma g(\phi(z))\phi'(z)\, dz$, so for such partitions

$$\left| \int_{\phi \circ \gamma} g(w)\, dw - \int_\gamma g(\phi(z))\phi'(z)\, dz \right| \leqslant \tfrac{\epsilon}{3} + \sum_{j=1}^n \varepsilon(t_{j-1}, t_j) + \tfrac{\epsilon}{3} = \tfrac{2\epsilon}{3} + \frac{4M\epsilon}{B} \sum_{j=1}^n |\gamma(j) - \gamma(t_{j-1})| \leqslant \tfrac{2\epsilon}{3} + \frac{4M\epsilon|\gamma|}{B}.$$

where $|\gamma|$ is the arclength of γ. Taking $B := 3 \cdot 4M|\gamma|$ above yields an upper bound of ϵ, and as $\epsilon > 0$ was arbitrary, the claim is proven.

Exercise 29

Let U be a non-empty open subset of $\mathbb{C}$. We show that the following statements are equivalent:

(i) U is topologically connected (i.e. only two clopen sets w.r.t. subspace topology of U are $\varnothing$ and U itself).

(ii) U is path connected (i.e. for every $z_1, z_2 \in U$ there is a curve γ whose image lies completely in U s.t. its initial point is z_1 and terminal point is z_2).

(iii) U is polygonally path connected (i.e. path connected as in (ii) but γ must now be a polygonal path)

Obviously we have (iii) $\implies$ (ii), so we show (ii) $\implies$ (i), and (i) $\implies$ (iii). First we have the following lemma: the unit interval $I := [0,1]$ is connected. *Proof:* suppose we have $I = A \cup B$ for non-empty disjoint open sets $A, B \subseteq \mathbb{R}$. Let $a \in A, b \in B$ and w.l.o.g. $a < b$. Define $s := \sup\{x \in \mathbb{R} : [a, x) \subseteq A\}$. Then $s \in (a, b) \subseteq I$; we must have $s > a$ because A is open so there is $[a, a+\epsilon) \subseteq A$, and $s < b$ because B is open means $(b-\epsilon, b+\epsilon) \subseteq B$. Well s must by in either A or B. If $s \in A$, then $(s-\epsilon, s+\epsilon) \subseteq A$, contradicting definition of sup. If $s \in B$, then $(s-\epsilon, s+\epsilon) \subseteq B$ by openness of B, and because A, B are disjoint, $[a, s-\frac{\epsilon}{2}) \not\subseteq A$, contradiction.

To show (ii) $\implies$ (i), suppose for sake of contradiction that U is not topologically connected. Then, there is some nonempty open set $A \subseteq U$ s.t. $B := U \setminus A$ is open (both open w.r.t. subspace topology of U in $\mathbb{C}$. But because U is open, A, B are in fact both open in $\mathbb{C}$ as well (since the subspace topology on $S \subseteq X$ for a topological space X is precisely the set of sets $S \cap O$ for O open in X, and S being open in X means $S \cap O$ is open in X). Then, taking $z_1 \in A, z_2 \in B$, we have that $\gamma^{-1}(A), \gamma^{-1}(B)$ are open (in subspace topology of $[0,1] \subseteq \mathbb{R}$), complements of each other, cover $[0,1]$ entirely (because $\text{im}(\gamma) \subseteq U = A \cup B$) and contain $0, 1$ respectively. But this is a contradiction because we have that $\gamma^{-1}(A) \subseteq [0,1]$ is a non-empty set that is both open and closed w.r.t. the subspace topology on $[0,1]$, which is impossible because we showed that $[0,1]$ is connected!

To show (i) $\implies$ (iii), we do as the hint suggests and fix a base point $z_0 \in U$ and consider the set S of all $z \in U$ that can be reached from z_0 by a polygonal path. First we show that S must be open. Suppose $z_1 \in U$ is reached via a polygonal path $[z_0 \rightsquigarrow z_1]$. Then because U is open, there is some $r > 0$ s.t. $B(z_1, r) \subseteq U$. But every $z \in B(z_1, r)$ can be reached via a polygonal path from z_0, namely the polygonal path $[z_0 \rightsquigarrow z_1]$ concatenated with the line segment $[z_1 \to z] \subseteq B(z_1, r) \subseteq U$ (it lies in the ball because balls are convex). Thus, we've shown $z_1 \in S \implies B(z_1, r) \subseteq S$, so S is open.

Now S must also be closed. Suppose we have a sequence $\{z_n\}_{n=1}^{\infty} \subseteq U$ that converge to some $z \in U$ where all the $z_n \in S$. I claim that $z \in S$ as well. By openness of U, there is ball $B(z, r) \subseteq U$. Since $z_n \to z$, there must be infinitely many $z_n \in B(z, r)$ (indeed for all $n \geqslant N$ for some $N \in \mathbb{N}$). Then because $z_n \in S$, there is a polygonal path $[z_0 \rightsquigarrow z_n]$, and again concatenating with the straight line segment $[z_n \to z] \subseteq B(z_1, r) \subseteq U$ (again using convexity of balls) yields a polygonal path $[z_0 \rightsquigarrow z]$. Thus, S is clopen in U and non-empty (contains z_0), so $S = U$ by topological connectedness.

Exercise 32

Let U be a non-empty connected open subset of $\mathbb{C}$, and let $f : U \to \mathbb{C}$ be continuous. We are asked to show that the following five conditions are equivalent:

(i) f possesses at least one antiderivative $F : U \to \mathbb{C}$.

(ii) $\oint_\gamma f(z)\,dz = 0$ (i.e. "conservativity") for all closed polygonal paths/curves in U.

(iii) $\oint_\gamma f(z)\,dz = 0$ for all *simple* closed polygonal paths/curves in U.

(iv) $\oint_\gamma f(z)\,dz = 0$ for all closed contours (i.e. concatenations of finitely many $\mathcal{C}^1$ paths) in U.

(v) $\oint_\gamma f(z)\,dz = 0$ for all closed rectifiable paths/curves in U.

Remark: note that the language used in (i) "f possesses *at least* one antiderivative F" suggests that there could be a wide class of antiderivatives of f, indeed it is the case that two antiderivatives F_1, F_2 of f must only differ by a constant. This is because defining $G := F_1 - F_2$, we have that G is complex differentiable hence Fréchet differentiable with derivative $G' = F_1' - F_2' = f - f = 0$ on all of U; and because any two points of U can be connected by a polygonal path, the (contrapositive of the) mean value theorem on the directional derivatives (which are linear combinations of the x, y-partials by Fréchet differentiability, i.e. all 0) tells us that indeed G must take the same value, say a constant $C \in \mathbb{C}$, at all points of U, which of course gives that $F_1 = F_2 + C$.

Outline: obviously we have (v) $\implies$ (iv) $\implies$ (ii) $\implies$ (iii) (since the class of curves considered in each condition in that chain is decreasing, i.e. simple closed polygonal paths $\subseteq$ closed polygonal paths $\subseteq$ closed contours $\subseteq$ closed rectifiable paths). FTC(dI) (Theorem 31 in Notes 2) gives (ii) $\implies$ (i), and FTC(Id) gives (i) $\implies$ (v). Thus, the only remaining piece to show is (iii) $\implies$ (ii). As the hint suggests, we proceed by induction on the number of edges in the closed polygonal paths.

We will write an arbitrary closed polygonal path as $[z_1 \to z_2 \to \ldots \to z_n \to z_1]$, and this number n will be called the *length* (different from "arclength") of the polygonal path. Paths of length one are precisely the constant functions $[a, b] \to \{z_1\}$, and from the definition of $\oint_{[z_1]} f(z)\,dz$ as a limit of Riemann sums, all such Riemann sums are exactly 0, so the integral is 0. For paths of length two $[z_1 \to z_2 \to z_1] = [z_1 \to z_2] + [z_2 \to z_1]$ (the equality comes from paragraph before Def. 6 in Notes 2), we have that $[z_2 \to z_1] \equiv -[z_1 \to z_2]$ (see paragraphs following Ex. 2 in Notes 2), so by Ex. 17(i)&(ii)&(iii) we have that $\oint_{[z_1 \to z_2 \to z_1]} f(z)\,dz = \int_{[z_1 \to z_2]} f(z)\,dz - \int_{[z_1 \to z_2]} f(z)\,dz = 0$.

Paths of length three $\gamma_3 := [z_1 \to z_2 \to z_3 \to z_1]$ form the base case for the induction we plan to do. If z_1, z_2, z_3 are not co-linear, then γ_3 is a simple curve, so condition (iii) tells us that $\oint_{\gamma_3} f(z)\,dz = 0$. If z_1, z_2, z_3 are co-linear, then ordering their real parts we get one of the $\mathfrak{Re}(z_i)$ is (using weak inequalities) between the other two (if all their real parts are the same, we can use imaginary parts instead; if all their imaginary parts are ALSO the same, then indeed $z_1 = z_2 = z_3$ and the path is a constant path, which we already dealt with in the previous paragraph).

A rigorous proof that for non-colinear (hence all distinct) z_1, z_2, z_3, the curve γ_3 is a simple is as follows: every point of γ_3 is weighted average of two of $z_{1,2,3}$. Thus suppose $tz_i + (1-t)z_j = sz_i + (1-s)z_k$. If $j = k$, then $(t-s)z_i + (s-t)z_j = 0 \implies (t-s)(z_i - z_j) \implies t = s$ by integral domain and distinctness of $z_{1,2,3}$. Otherwise $j \neq k$. If $s = 1$, then $(1-t)z_j = (1-t)z_i \implies t = 1$ by cancellation property in integral domain and distinctness of $z_{1,2,3}$. If $s \neq 1$, then $z_k = \frac{t-s}{1-s} z_i + \frac{1-t}{1-s} z_j = \frac{1-t}{1-s}(z_j - z_i) + \frac{1-s}{1-s} z_i$, i.e. z_k is on the line $z_i + \lambda(z_j - z_i)$, contradicting non-colinearity. *Remark*: **gold** color indicates "written after having turned in the homework".

Let us denote z_j to be the one "between the other two" in the sense described above. Then $[z_{j-1} \to z_j] + [z_j \to z_{j+1}]$ is a reparameterization of $[z_{j-1} \to z_{j+1}]$ (where indices are taken modulo 3 and lie in $\{1, 2, 3\}$). This can be seen by finding the explicit formula for $[z_{j-1} \to z_j] + [z_j \to z_{j+1}]$ from $[0, 2] \to \mathbb{C}$ (assuming all the individual line segments we've been dealing with are parameterized by $[0, 1]$) from the definition of concatenation (see paragraphs following Ex. 2 in Notes 2), and then constructing a (piecewise linear) monotone function ϕ between $[0, 2]$ and $[0, 1]$ (on which $[z_{j-1} \to z_{j+1}]$ is defined) s.t. $[[z_{j-1} \to z_j] + [z_j \to z_{j+1}]](t) = [z_{j-1} \to z_{j+1}](\phi(t))$.

Because $\mathfrak{Re}(z_j)$ is between $\mathfrak{Re}(z_{j-1})$ and $\mathfrak{Re}(z_{j+1})$ (or again if the real parts are all the same we can deal with imaginary parts instead) it is a weighted average of the two, say of weight $s \in [0, 1]$. Since we assumed co-linearity we know that z_j is on the line $(1-t)z_{j-1} + tz_{j+1}$, $t \in \mathbb{R}$ so comparing real parts yields $(1-s)z_{j-1} + sz_{j+1} = z_j$. Now we construct $\phi : [0, 2] \to [0, 1]$. If $t \in [0, 1]$, want $\phi(t)$ s.t. $\phi(0) = 0, \phi(1) = s$ and $(1-t)z_{j-1} + tz_j = (1-\phi(t))z_{j-1} + \phi(t)z_{j+1}$. Intuitively $\phi(t)$ is linear so we check $\phi(t) = ts$ works: $(1-st)z_{j-1} + stz_{j+1} = (1-st)z_{j-1} + (tz_j - t(1-s)z_{j-1}) = tz_j + (1-t)z_{j-1}$ as desired. If $t \in [1, 2]$, $\phi(1) = s, \phi(2) = 1$ and need $(2-t)z_j + (1-t)z_{j+1} = (1-\phi(t))z_{j-1} + \phi(t)z_{j+1}$. Again intuitively ϕ linear so take $\phi(t) = s + (1-s)(1-t)$. Check: $(1-(s+(1-s)(t-1)))z_{j-1} + (s+(1-s)(t-1))z_{j+1} = ((1-s)(1-(t-1)))z_{j-1} + (s(1-(t-1)) + (t-1))z_{j+1} = (2-t)((1-s)z_{j-1} + sz_{j+1}) + (t-1)z_{j+1} = (2-t)z_j + (t-1)z_{j+1}$ as desired. It is now clear that this function ϕ we have defined on $[0, 2]$ to satisfy $[[z_{j-1} \to z_j] + [z_j \to z_{j+1}]](t) = [z_{j-1} \to z_{j+1}](\phi(t))$ (i.e. the two conditions we checked/computed above in turqoise) is indeed monotone and has $\phi(0) = 0, \phi(2) = 1$ and so by Ex. 2(iii) we are done.

Again using $[z_{j-1} \to z_{j+1}] \equiv -[z_{j+1} \to z_{j-1}]$ (see paragraphs following Ex. 2), the definition of polygonal curves in the paragraph before Def. 6, Ex. 17(i)&(ii)&(iii), and the previous paragraph, we get

$$\begin{aligned}\oint_{[z_{j-1} \to z_j \to z_{j+1} \to z_{j-1}]} &= \int_{[z_{j-1} \to z_j]} + \int_{[z_j \to z_{j+1}]} + \int_{[z_{j+1} \to z_{j-1}]} \\ &= \int_{[z_{j-1} \to z_j] + [z_j \to z_{j+1}]} + \int_{[z_{j+1} \to z_{j-1}]} = -\int_{[z_{j+1} \to z_{j-1}]} + \int_{[z_{j+1} \to z_{j-1}]} = 0\end{aligned}$$

(the "$f(z)\,dz$" following the integral signs are implied). We abbreviate the aforementioned three key pieces of information (line segment reversal in paragraphs following Ex. 2, definition of polygonal curves in paragraph before Def. 6, and Ex. 17(i)&(ii)&(iii)) by **KEY FACTS**.

Since the polygonal/triangular path $[z_1 \to z_2 \to z_3 \to z_1]$ is defined (see the paragraph before Def. 6 in Notes 2) to be the concatenation $[z_1 \to z_2] + [z_2 \to z_3] + [z_3 \to z_1]$, we have by Ex. 17(iii) that $\oint_{[z_1 \to z_2 \to z_3 \to z_1]} = \int_{[z_1 \to z_2]} + \int_{[z_2 \to z_3]} + \int_{[z_3 \to z_1]}$. By commutativity of addition for complex numbers, we can cyclically permute those 3 individual integrals, and then use Ex. 17(iii) again to get that indeed $\oint_{[z_1 \to z_2 \to z_3 \to z_1]} = \oint_{[z_j \to z_{j+1} \to z_{j+2} \to z_j]}$ for any $j \in \{1, 2, 3\}$ (where again indices are taken modulo 3 and lie in $\{1, 2, 3\}$). This in conjunction with the big display equation gives that $\oint_{[z_1 \to z_2 \to z_3 \to z_1]} f(z)\, dz = 0$, as desired.

Finally, the induction step. Given a polygonal path $[z_1 \to \ldots \to z_n \to z_1]$, we use the **KEY FACTS** from above to write

$$
\begin{aligned}
\oint_{[z_1 \to \ldots \to z_n \to z_1]} &= \int_{[z_1 \to z_2]} + \ldots + \int_{[z_{n-1} \to z_n]} + \int_{[z_n \to z_1]} \\
&= \int_{[z_1 \to z_2]} + \ldots + \int_{[z_{n-1} \to z_1]} - \int_{[z_{n-1} \to z_1]} + \int_{[z_{n-1} \to z_n]} + \int_{[z_n \to z_1]} \\
&= \int_{[z_1 \to z_2]} + \ldots + \int_{[z_{n-1} \to z_1]} + \int_{[z_1 \to z_{n-1}]} + \int_{[z_{n-1} \to z_n]} + \int_{[z_n \to z_1]} \\
&= \oint_{[z_1 \to \ldots \to z_{n-1} \to z_1]} + \oint_{[z_1 \to z_{n-1} \to z_n \to z_1]} = 0 + 0 = 0
\end{aligned}
$$

where the last line of equalities is from the induction hypothesis ($[z_1 \to \ldots \to z_{n-1} \to z_1]$ is a closed polygonal path of length $n - 1$, and $[z_1 \to z_{n-1} \to z_n \to z_1]$ is a closed polygonal path of length 3).

246A Homework 2

Daniel Rui - 10/15/21

The problems titled with "Exercise" come from Terry Tao's Notes 1, and the problem titled with "Exercise S&S" come from Stein & Shakarchi (the number after the § symbol is the chapter number, and the number after the period is the exercise number).

Exercise 9

Let $\sum_{n=0}^{\infty} a_n(z-z_0)$ be a formal power series with all the a_n non-zero for all sufficiently large n. We show that the radius of convergence R of the series obeys the bounds $\liminf_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} \leqslant R \leqslant \limsup_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$ (and give an example in which both inequalities are strict).

Recall that Proposition 7 of Notes 1 tells us that for all formal power series centered at z_0, there is a number $R \in [0,\infty]$ (the "radius of convergence") for which the series is absolutely convergent everywhere in the interior of the ball/disk $B(z_0, R)$, and divergent everywhere in the exterior of the ball (behavior on the boundary unspecified), where furthermore it turns out that R is precisely equal to $\liminf_{n\to\infty} |a_n|^{-1/n} = {}^1\!/\limsup_{n\to\infty} |a_n|^{1/n}$ (Cauchy-Hadamard formula).

Philosophical Interlude

Let us discuss briefly the intuition behind the Cauchy-Hadamard formula, and the root/ratio tests in general. The basic idea is that Cauchy-Hadamard can be "grown" from the "seed" of "applying the (absolute) convergence/divergence of the geometric series to $\sum a_n z^n = \sum (a_n^{1/n} z)^n$". In other words, if we have $r \in (0,1)$ s.t. $|a_n^{1/n} z| \leqslant r \iff |z| \leqslant r|a_n|^{-1/n}$ for all (sufficiently large) n, then we get absolute convergence of $\sum a_n z^n$; and if there is $r > 1$ s.t. $|a_n^{1/n} z| \geqslant r \iff |z| \geqslant r|a_n^{-1/n}|$ for infinitely many n, then $\sum a_n z^n$ diverges. This "for all sufficiently large n/for infinitely many n" dichotomy between the convergent/divergence conditions are highly reminiscent (or "reek", so to speak) of liminf, and indeed this suggestion can be (quite straightforwardly) made rigorous, as in Prop. 7 of Notes 1.

The ratio test seems a bit hard to intuit from this perspective (where/how/why do two terms of the sequence come in?), but if we instead have the perspective that a geometric series is precisely one where the $(n+1)$th term is the nth term scaled down by $r \in (0,1) \iff |\frac{a_{n+1}}{a_n}| = r \in (0,1)$, the ratio test essentially tells us that for "approximate geometric" or indeed "approximate subgeometric" series — i.e. where $|\frac{a_{n+1}}{a_n}| \lessapprox r \in (0,1)$ for only potentially large n since of course we only care about tail behavior (this is formalized as $\limsup_{n\to\infty} |\frac{a_{n+1}}{a_n}| = r \in (0,1)$) — we get absolute convergence. Applying this to $\sum a_n z^n$, we are basically saying that $\limsup_{n\to\infty} |\frac{a_{n+1}}{a_n}||z| = r \in (0,1) \iff |z| < \liminf_{n\to\infty} |\frac{a_n}{a_{n+1}}|$ implies absolute convergence. This is a "hand-wavy proof" that $\liminf_{n\to\infty} |\frac{a_n}{a_{n+1}}| \leqslant R$.

Proof of Exercise

We focus on the left inequality $r := \liminf_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} \leqslant R = \liminf_{n\to\infty} |a_n|^{-1/n}$ first.

Direct approach: if $r = 0$, the claim is trivial. Otherwise, $r \in (0, \infty]$. Let $\ell \in (0, \infty)$ be an arbitrary lesser number $\ell < r$ (this way we handle both $r \in (0, \infty)$ and $r = \infty$ cases at the same time). By the definition of liminf, there is $N \in \mathbb{N}$ s.t. $n \geqslant N \implies \frac{|a_n|}{|a_{n+1}|} > \ell$. Therefore we have that $n \geqslant N \implies \frac{|a_N|}{|a_n|} = \frac{|a_N|}{|a_{N+1}} \frac{|a_{N+1}|}{|a_{N+2}|} \cdots \frac{|a_{N+(n-N-1)}|}{|a_{N+(n-N)}|} > \ell^{n-N}$, or dividing by $|a_N|$ and applying $\bullet^{1/n}$ on both sides (which preserves inequalities because it's a strictly increasing function), we have $n \geqslant N \implies |a_n|^{-1/n} > \ell \cdot (\frac{1}{\ell^N |a_N|})^{1/n}$.

As $C := (\frac{1}{\ell^N |a_N|})$ is just some positive constant (recall $\ell \in (0, \infty)$ and we are assuming the a_n are non-zero for sufficiently large n), and the function $C^x = \exp(x \ln(C))$ is continuous on $\mathbb{R}$, we have $n \to \infty \implies \frac{1}{n} \to 0 \implies C^{1/n} \to 1$. Thus taking the liminf of both sides of $|a_n|^{-1/n} > \ell \cdot (\frac{1}{\ell^N |a_N|})^{1/n}$ yields $R \geqslant \ell \cdot 1$. As $\ell < r$ was arbitrary, we indeed have that $R \geqslant r$ (again this works for both finite and infinite r).

For the right inequality $R \leqslant \limsup_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} =: r'$, this is trivial if $r' = \infty$ so we can now consider only $r' < \infty$. For any arbitrary greater real number $g > r'$, by the definition of limsup, there is $N \in \mathbb{N}$ s.t. $n \geqslant N \implies \frac{|a_n|}{|a_{n+1}|} < g$. Analogous to above, this shows that $n \geqslant N \implies |a_n|^{-1/n} < g \cdot (\frac{1}{g^N |a_N|})^{1/n}$. Taking limsup of both sides (and using the above discussion on $C^{1/n} \to 0$ for the constant $C := (\frac{1}{g^N |a_N|})$) yields $\limsup_{n\to\infty} |a_n|^{-1/n} \leqslant g$. As $g > r'$ was arbitrary, we indeed have $R \leqslant \limsup_{n\to\infty} |a_n|^{-1/n} \leqslant r'$.

Example of Strict Inequality

Take the sequence of a_n to be 1 for n odd and $\frac{1}{2}$ for n even. Then the ratios $|\frac{a_n}{a_{n+1}}|$ are 2 for n odd and $\frac{1}{2}$ for n even, and so the liminf and limsup values of the ratios are respectively $\frac{1}{2}$ and 2. But $R := \liminf_{n\to\infty} |a_n|^{-1/n} = 1/\limsup_{n\to\infty} |a_n|^{1/n}$, where $\limsup_{n\to\infty} |a_n|^{1/n} = 1$ since $1^{1/n} = 1$ for all $n \in \mathbb{N}$, and $(\frac{1}{2})^{1/n} < 1$ for all $n \in \mathbb{N}$ (in fact $\lim_{n\to\infty} (\frac{1}{2})^{1/n} = 1$ since it is a monotone increasing sequence bounded by 1 and hence has a limit L, and $L = 1$ because if $L < 1$, L^n would be very close to 0, not $\frac{1}{2}$; this all implies that $\lim_{n\to\infty} |a_n|^{1/n} = 1$). And indeed we have $\frac{1}{2} < 1 < 2$.

Exercise 14

This exercise we use the summation by parts formula to ultimately prove Abel's theorem.

(a) Let $a_0, \ldots, a_N$ be a finite sequence of complex numbers, and let $A_n := a_0 + \ldots + a_n$ be the partial sums for $n \in \{0, \ldots, N\}$. Then, for any complex numbers $b_0, \ldots, b_N$ (denoting $A_{-1} = 0$):

$$\begin{aligned}\sum_{n=0}^{N} a_n b_n &= \sum_{n=0}^{N} (A_n - A_{n-1}) b_n = \sum_{n=0}^{N} A_n b_n - \sum_{n=0}^{N} A_{n-1} b_n \\ &= \sum_{n=0}^{N} A_n b_n - \sum_{n=1}^{N} A_{n-1} b_n = \sum_{n=0}^{N} A_n b_n - \sum_{n=0}^{N-1} A_n b_{n+1} \\ &= \sum_{n=0}^{N-1} A_n (b_n - b_{n+1}) + A_N b_N.\end{aligned}$$

(b) Let $a_0, \ldots,$ be a sequence of complex numbers s.t. $A := \sum_{n=0}^{\infty} a_n$ is convergent (but not necessarily absolutely). The problem statement takes $A = 0$, but we prove the claim for any $A \in \mathbb{R}$. We show that $\sum_{n=0}^{\infty} a_n r^n$ is absolutely convergent for any $r \in (0,1)$, and that $\lim_{r\to 1^-} \sum_{n=0}^{\infty} a_n r^n = A$. First, Prop. 7 from Notes 1 tells us that for any power series such as $\sum_{n=0}^{\infty} a_n z^n$, there is some R s.t. $|z| < R \implies$ the series is absolutely convergent, and $|z| > R \implies$ the series is divergent. Because $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} a_n (1)^n$ is convergent, it can not be that $R < 1$ (since $|1| > R$ would imply the series diverges, which it does not), so $R \geqslant 1$, implying that $r \in (0,1) \implies |r| < R \implies \sum_{n=0}^{\infty} a_n r^n$ is absolutely convergent.

We now prove the statement about the limit. Let us fix $|z| < 1$. Denote $S_N(z) = \sum_{n=0}^{N} a_n z^n$, $S(z) = \sum_{n=0}^{\infty} a_n z^n$, and $A_N = \sum_{n=0}^{N} a_n$. By (a) where $b_n = z^n$, we have $S_n(z) = A_N z^N + \sum_{n=0}^{N-1} A_n(z^n - z^{n+1}) = A_N z^N + (1-z)\sum_{n=0}^{N-1} A_n z^n$. Then,

$$\begin{aligned}
|S_N(z) - A| &= \left| A_N z^N + (1-z)\sum_{n=0}^{N-1} A_n z^n - (1-z)\frac{A}{1-z} \right| \\
&= \left| A_N z^N + (1-z)\sum_{n=0}^{N-1} A_n z^n - (1-z)\left(\sum_{n=0}^{N-1} A z^n + \sum_{n=N}^{\infty} A z^n \right) \right| \\
&= \left| A_N z^N + (1-z)\sum_{n=0}^{N-1} (A_n - A) z^n - (1-z)\frac{A z^N}{1-z} \right| \\
&\leqslant |A_N - A||z|^N + |1-z|\sum_{n=0}^{N-1} |A_n - A||z|^n.
\end{aligned}$$

Taking the limit as $N \to \infty$ and using that $|z| < 1$, we get $|S(z) - A| \leqslant |1-z|\sum_{n=0}^{\infty} |A_n - A||z|^n$ (and this is for all $|z| < 1$). Now fix $\epsilon > 0$. By convergence $A_n \to A$, there is $N \in \mathbb{N}$ s.t. $n \geqslant N \implies |A_n - A| < \frac{\epsilon}{2}$. For $n < N$, A_n, A are all bounded, so there is some $B \in \mathbb{N}$ s.t. $|A_n - A| \leqslant B$ for all $n < N$. Then, for all $|z| < 1$ we have

$$\begin{aligned}
|S(z) - A| &\leqslant |1-z|\left(\sum_{n=0}^{N-1} |A_n - A||z|^n + \sum_{n=N}^{\infty} |A_n - A||z|^n \right) \\
&\leqslant |1-z|\left(\sum_{n=0}^{N-1} B \cdot 1^n + \sum_{n=N}^{\infty} \frac{\epsilon}{2}|z|^n \right) = |1-z|\left(BN + |z|^N \frac{\epsilon/2}{1-|z|} \right) \\
&= BN|1-z| + 1^N \frac{\epsilon/2}{1-|z|}|1-z|.
\end{aligned}$$

Setting $z = r \in (0,1)$, we get that $|S(r) - A| \leqslant BN(1-r) + \frac{\epsilon}{2}$. As BN is just a constant, there is $\delta > 0$ s.t. $BN\delta < \frac{\epsilon}{2}$, and so then $r \in (1-\delta, 1) \implies |S(r) - A| \leqslant BN(1-r) + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$, which is precisely what it means for $\lim_{r\to 1^-} S(x) = A$, as desired.

(c) (Abel's theorem) Let $F(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ be a power series with finite positive radius of convergence R, and let $z_1 := z_0 + Re^{i\theta}$ be a point on the boundary of the disk of convergence at which the series $\sum_{n=0}^{\infty} a_n (z_1 - z_0)^n$ converges. We show that $\lim_{r\to R^-} F(z_0 + re^{i\theta}) = F(z_1)$.

Define $G(z) = F(Re^{i\theta}z + z_0) = \sum_{n=0}^{\infty} a_n R^n e^{i\theta n} z^n$. Then the power series of G has radius of convergence $\liminf_{n\to\infty} |a_n R^n e^{i\theta n}|^{-1/n} = R^{-1} \liminf_{n\to\infty} |a_n|^{-1/n} = R^{-1} \cdot R = 1$ (using Prop. 7 in Notes 1), and $G(1) = F(z_1)$ converges. Part (b) applied to this situation now gives us that $\lim_{x\to 1^-} G(x) = G(1) = F(z_1)$, but of course $F(z_1) = \lim_{x\to 1^-} G(x) = \lim_{x\to 1^-} F(Re^{i\theta}x + z_0) = \lim_{r\to R^-} F(re^{i\theta} + z_0)$, as desired.

Exercise 17

Recall that Theorem 15 of Notes 1 says that for $F(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ with positive radius of convergence R, F is holomorphic (i.e. complex differentiable everywhere) on the disk of convergence $D(z_0, R)$, and the derivative has power series $F'(z) = \sum_{n=0}^{\infty}(n + 1)a_{n+1}(z - z_0)^n$ with the same radius of convergence R.

By induction, one has that the mth derivative $F^{(m)}(z) = \sum_{n=0}^{\infty}(n+1)(n+2)\cdots(n+m)a_{n+m}(z-z_0)^n$ with radius of convergence (r.o.c.) R (since if we know $F^{(m)}(z) = \sum_{n=0}^{\infty}(n+1)(n+2)\cdots(n+m)a_{n+m}(z-z_0)^n$ with r.o.c. R, then applying Theorem 15 gives $F^{(m+1)}(z) = \sum_{n=0}^{\infty}(n + 1)(n + 1 + 1)(n + 2 + 1)\cdots(n + m + 1)a_{n+m+1}(z - z_0)^n$ with r.o.c. R, as expected). Because this is true for all $z \in D(z_0, R)$, we can in particular set $z = z_0$, which yields $F^{(m)}(z_0) = (0 + 1)(0 + 2)\cdots(0 + m)a_{0+m} = m!a_m$ (for higher powers $n \geqslant 1$, $(z - z_0)^n = 0$, so the only term in the infinite sum is indeed the 0th term). Thus $a_n = \frac{1}{n!}F^{(n)}(z_0)$ for all $n \in \mathbb{N}$.

In particular, if we have another power series $G(z) = \sum_{n=0}^{\infty} b_n(z - z_0)^n$ with a positive radius of convergence R' that agrees with F on some neighborhood U of z_0, then the above paragraph tells us that $b_n = \frac{1}{n!}G^{(n)}(z_0)$, but since F, G agree on a neighborhood U of z_0 and one only needs local information at z_0 (i.e. values on *some* neighborhood of z_0) to determine all order derivatives, we have that $b_n = \frac{1}{n!}G^{(n)}(z_0) = \frac{1}{n!}F^{(n)}(z_0) = a_n$ for all $n \geqslant 0$ (i.e. we have proven uniqueness of power series).

Remarks: Marshall (in Theorem 2.8) proves this theorem directly, i.e. without differentiation, and his proof has the added benefit of showing that if f is analytic at z_0 (so $f = \sum c_n(z - z_0)^n$ near z_0), then either all the $c_n = 0$ (f is identically 0 near z_0) or some $c_n \neq 0$, in which case there is some $\delta > 0$ s.t. f is definitely not zero on $0 < |z - z_0| < \delta$. In other words, if f is analytic on a connected open Ω (i.e. at every $z_0 \in \Omega$, f can be written locally as a power series centered at z_0), then either f is identically 0 on Ω (can show that if $c_n = 0$ for all $n \in \mathbb{N}$ for just one power series, then the set S of all $z_0 \in \mathbb{C}$ at which the power series centered at z_0 have all coefficients 0 is non-empty, open, and closed i.e. contains its boundary), or otherwise the zeroes of f are isolated (since we are assuming all local power series representations of f do not have all non-zero coefficients, the above dichotomy tells us that every zero z_0 has δ-ball around z_0 on which f is never zero other than at z_0). This yields the famous identity theorem, which tells us that if two analytic functions (on some open connected Ω) agree on a set with an accumulation point, then $f = g$ on ALL of Ω.

Exercise 23

This exercise defines and examines the Wirtinger derivatives. Let $U \subseteq \mathbb{C}$ be open, and $f : U \to \mathbb{C}$ be a Fréchet differentiable function. We define the Wirtinger derivatives $\partial_z f := \frac{\partial f}{\partial z} : U \to \mathbb{C}$ and $\partial_{\overline{z}} f := \frac{\partial f}{\partial \overline{z}} : U \to \mathbb{C}$ by the formulae

$$\partial_z f := \tfrac{1}{2}(\partial_x f + \tfrac{1}{i}\partial_y f) \quad \text{and} \quad \partial_{\overline{z}} f = \tfrac{1}{2}(\partial_x f - \tfrac{1}{i}\partial_y f).$$

Be aware that sometimes I abbreviate $\frac{\partial f}{\partial x}$ by $\partial_x f$, or even f_x.

(i) We show that f is holomorphic on U if and only if the Wirtinger derivative $\partial_{\overline{z}} f$ vanishes identically on U. Recall from class (or Prop. 20 in Notes 1) that a function is Fréchet differentiable at a point z_0 and satisfying the Cauchy-Riemann equations if and only if f is holomorphic (i.e. complex differentiable) at z_0. Thus it suffices to prove that f (as defined at the preamble of this exercise) satisfies the Cauchy-Riemann equations on all of Ω if and only if $\partial_{\overline{z}} f \equiv 0$ (i.e. vanishes identically) on Ω.

Recall from class (or Prop. 20) that the Cauchy-Riemann equation is precisely $\partial_x f = \frac{1}{i}\partial_y f$ (at the point z_0 of holomorphicity or on the region Ω of holomorphicity), and indeed $\partial_x f = \frac{1}{i}\partial_y f$ on all of $\Omega \iff \partial_{\overline{z}} f = \frac{1}{2}(\partial_x f - \frac{1}{i}\partial_y f) = 0$ on all of Ω, as desired.

(ii) If f is given by the polynomial $f(z) = \sum_{n,m\geqslant 0, n+m\leqslant d} c_{n,m} z^n \overline{z}^m$ for some $c_{n,m} \in \mathbb{C}$ and $d \in \mathbb{N}$, we want to show that

$$[\partial_z f](z) = \sum_{n,m\geqslant 0, n+m\leqslant d} c_{n,m}(n z^{n-1})\overline{z}^m \quad \text{and} \quad [\partial_{\overline{z}} f](z) = \sum_{n,m\geqslant 0, n+m\leqslant d} c_{n,m} z^n (m\overline{z}^{m-1}).$$

As the hint suggests, we first establish a Leibniz rule for the Wirtinger derivatives. For a product fg of Fréchet differentiable functions, we have $\partial_z(fg) := \frac{1}{2}(\partial_x(fg) + \frac{1}{i}\partial_y(fg)) = \frac{1}{2}([\partial_x f]g + f[\partial_x g] + \frac{1}{i}([\partial_y f]g + f[\partial_y g])) = [\partial_z f]g + f[\partial_z g]$. Essentially the same computation yields $\partial_{\overline{z}}(fg) = [\partial_{\overline{z}} f]g + f[\partial_{\overline{z}} g]$. Both Wirtinger derivatives are also very easily seen to be linear.

It is trivial to check for $z = x + iy$ that $\partial_z z = \frac{1}{2}(1 + \frac{i}{i}) = 1, \partial_{\overline{z}} z = \frac{1}{2}(1 - \frac{i}{i}) = 0$; and similarly $\partial_z \overline{z} = \frac{1}{2}(1 + \frac{-i}{i}) = 0, \partial_{\overline{z}} \overline{z} = \frac{1}{2}(1 - \frac{-i}{i}) = 1$. We can now prove easily by induction that $\partial_z z^n = n z^{n-1}$ (since $\partial_z z^{n+1} = \partial_z(z^n z) = [\partial_z z^n]z + z^n[\partial_z z] = (n z^{n-1}) \cdot z + z^n \cdot 1 = (n+1)z^n$); and similarly $\partial_{\overline{z}} \overline{z}^m = m\overline{z}^{m-1}$, and $\partial_{\overline{z}} z^n = \partial_z \overline{z}^m = 0$. Applying these facts and linearity, we get the desired formulae.

We conclude that f is holomorphic if and only if $c_{n,m} = 0$ for $m \geqslant 1$, since we know from part (a) that f is holomorphic iff $\partial_{\overline{z}} f \equiv 0$, and $\sum_{n,m\geqslant 0, n+m\leqslant d} c_{n,m} z^n (m\overline{z}^{m-1}) \equiv 0 \iff c_{n,m} = 0$ for $m \geqslant 1$. This last " $\iff$ " is from the **Fact** (we will reference this **Fact** later in Ex. 27 below) that we can take n derivatives ∂_z and m derivatives $\partial_{\overline{z}}$ (taking derivatives of the 0 on the RHS remains 0) and set $z = 0$ to get $c_{m,n} = 0$.

(iii) We want to show that f is totally/Fréchet differentiable (when viewed as a function $\mathbb{R}^2 \to \mathbb{R}^2$) at $z_0 \in U$ if and only if there are complex numbers a, b s.t. $f(z) = f(z_0) + a(z - z_0) + b(\overline{z - z_0}) + o(|z - z_0|)$ (where the o is w.r.t. the limit as $z \to z_0$; i.e. $o(|z - z_0|)$ represents a function ψ s.t. $\frac{\psi(z)}{|z-z_0|} \to 0 \iff \left|\frac{\psi(z)}{|z-z_0|} - 0\right| \to 0$ as $z \to z_0$). We will furthermore show that a, b are uniquely determined by the above equation, and satisfy $a = \frac{1}{2}(f_x - if_y)$ and $b = \frac{1}{2}(f_x + if_y)$.

Let us write $z_0 = x_0 + iy_0$. Recall that the definition of total/Fréchet differentiability at $(x_0, y_0) \in U \subseteq \mathbb{R}^2$ is that

$$f((x, y)) = f((x_0, y_0)) + L((x - x_0, y - y_0)) + r((x - x_0, y - y_0))$$

where the remainder $r : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies $\lim_{(x,y)\to(x_0,y_0)} \frac{|r((x-x_0,y-y_0))|}{|(x-x_0,y-y_0)|} = 0$, and where L is a $\mathbb{R}$-linear function, i.e. $L((x - x_0, y - y_0)) = (\alpha(x - x_0) + \beta(y - y_0), \alpha'(x - x_0) + \beta'(y - y_0))$ for some $\alpha, \alpha', \beta, \beta' \in \mathbb{R}$. The condition on r makes it clear that $r(z - z_0)$ (now thinking of r as a function $\mathbb{C} \to \mathbb{C}$) is in fact $o(|z - z_0|)$. Thus, to prove the claim, all we need to do is show that L can be written as $a(z - z_0) + b(\overline{z - z_0})$ for some $a, b \in \mathbb{C}$ and that any $a(z - z_0) + b(\overline{z - z_0})$ (after converting from $\mathbb{C}$ to $\mathbb{R}^2$) is an $\mathbb{R}$-linear function.

Suppose we are given $a, b \in \mathbb{C}$. Let us write $a = a_x + ia_y$ and $b = b_x + ib_y$ for $a_x, a_y, b_x, b_y \in \mathbb{R}$. Then, $a(z - z_0) + b(\overline{z - z_0}) = a_x(x - x_0) - a_y(y - y_0) + ia_x(y - y_0) + ia_y(x - x_0) + b_x(x - x_0) + b_y(y - y_0) - ib_x(y - y_0) + ib_y(x - x_0)$. Simplifying and converting from $\mathbb{C}$ to $\mathbb{R}^2$, it is easy to see that this will be of the form $(\alpha(x - x_0) + \beta(y - y_0), \alpha'(x - x_0) + \beta'(y - y_0))$.

For the other direction, $L((x - x_0, y - y_0)) = (\alpha(x - x_0) + \beta(y - y_0), \alpha'(x - x_0) + \beta'(y - y_0))$ thought of as a complex number is $\alpha(x - x_0) + \beta(y - y_0) + i\alpha'(x - x_0) + i\beta'(y - y_0)$. But $x - x_0 = \frac{1}{2}(z - z_0 + \overline{z - z_0})$, and $y - y_0 = -\frac{1}{2}i(z - z_0 - \overline{z - z_0})$. Substituting these identities in and simplifying, we will get that $L((x - x_0, y - y_0)) = a(z - z_0) + b(\overline{z - z_0})$ for some $a, b \in \mathbb{C}$.

As for uniqueness, if $f(z) = f(z_0) + a(z - z_0) + b(\overline{z - z_0}) + o(|z - z_0|) = f(z_0) + a'(z - z_0) + b'(\overline{z - z_0}) + o(|z - z_0|)$, then we get that $(a - a')(z - z_0) + (b - b')(\overline{z - z_0}) + o(|z - z_0|)$ (difference of two $o(|z - z_0|)$ functions is still $o(|z - z_0|)$ by the triangle inequality), or in other words $(a - a')(z - z_0) + (b - b')(\overline{z - z_0})$ is $o(|z - z_0|)$. Hence, we have $\lim_{z\to z_0} \frac{|(a-a')(z-z_0)+(b-b')(\overline{z-z_0})|}{|z-z_0|} = 0$. Now fixing $z \in \mathbb{C}$, the above limit being 0 means that the following limit ($\lambda \in \mathbb{R}$) is also 0: $\lim_{\lambda\to 0} \frac{|(a-a')(\lambda(z-z_0))+(b-b')(\overline{\lambda(z-z_0)})|}{|\lambda(z-z_0)|} = 0$. But this fraction is simply $\frac{|(a-a')(z-z_0)+(b-b')(\overline{z-z_0})|}{|z-z_0|}$, since we can move all the λ's in front by linearity! In other words, we have that for all $z \in \mathbb{C}$, $|(a - a')(z - z_0) + (b - b')(\overline{z - z_0})| = 0$, which forces $a - a' = 0 = b - b'$, as desired.

Finally, taking $z = z_0 + \lambda$ where $\lambda \in \mathbb{R}$, we have that $f(z) = f(z_0) + a\lambda + b\lambda + o(|\lambda|) \implies \lim_{\lambda\to 0} \frac{f(z)-f(z_0)}{\lambda} = a+b$. But this limit is by definition $f_x(z_0)$, so we have $f_x(z_0) = a+b$. Similarly, we can take $z = z_0 + \lambda i$ where again $\lambda \in \mathbb{R}$, to get that $f(z) = f(z_0) + a\lambda i - b\lambda i + o(|\lambda|) \implies f_y(z_0) = \lim_{\lambda\to 0} \frac{f(z)-f(z_0)}{\lambda} = i(a - b)$. Thus, indeed a, b satisfy $a = \frac{1}{2}(f_x(z_0) - if_y(z_0))$ and $b = \frac{1}{2}(f_x(z_0) + if_y(z_0))$.

Exercise 27

This exercise examines the relationship between the Wirtinger derivatives and harmonicity. See Exercise 23 above for notation. **Be aware** that I abbreviate $\frac{\partial f}{\partial z}$ by $\partial_z f$ sometimes.

(i) If $f : U \to \mathbb{C}$ is $\mathcal{C}^2$ on an open subset $U \subseteq \mathbb{C}$, we show that $\Delta f = 4\partial_z \partial_{\overline{z}} f = 4\partial_{\overline{z}} \partial_z f$. Recall the definition of the Laplacian $\Delta f := \partial_x \partial_x f + \partial_y \partial_y f$. We compute:

$$\begin{aligned}\partial_z \partial_{\overline{z}} f &= \tfrac{1}{2}[\partial_x \tfrac{1}{2}(\partial_x f - \tfrac{1}{i}\partial_y f)] + \tfrac{1}{2i}[\partial_y \tfrac{1}{2}(\partial_x f - \tfrac{1}{i}\partial_y f)] \\ &= \tfrac{1}{4}\partial_x \partial_x f - \tfrac{1}{4i}\partial_x \partial_y f + \tfrac{1}{4i}\partial_y \partial_x f - \tfrac{1}{4i^2}\partial_y \partial_y f = \tfrac{1}{4}(\partial_x \partial_x f + \partial_y \partial_y f) = \tfrac{1}{4}\Delta f\end{aligned}$$

where we used that for $\mathcal{C}^2$ functions f on U, Clairaut's theorem (see the paragraphs below Remark 24 in Notes 1) gives that $\partial_x \partial_y f = \partial_y \partial_x f$ everywhere on U. Essentially the same computation yields $\Delta f = 4\partial_{\overline{z}} \partial_z f$.

This yields an alternate route to the fact that $\mathcal{C}^2$ holomorphic functions are harmonic, because recall from Exercise 23 above that f is holomorphic on $U \iff \partial_{\overline{z}} f \equiv 0$ on U, so here if f is holomorphic, then $\Delta f = \partial_z[\partial_{\overline{z}} f] = \partial_z 0 = 0$ everywhere on U.

(ii) If $f = \sum_{n,m \geqslant 0, n+m \leqslant d} c_{n,m} z^n \overline{z}^m$ for some $c_{n,m} \in \mathbb{C}$ and $d \in \mathbb{N}$, we want to show that f is harmonic on $\mathbb{C}$ if and only if $c_{n,m} = 0$ if n, m are both positive (i.e. f contains only pure z or $\overline{z}$ terms, no blends). Recall from Exercise 23(ii) above that

$$[\partial_z f](z) = \sum_{n,m \geqslant 0, n+m \leqslant d} c_{n,m}(nz^{n-1})\overline{z}^m \quad \text{and} \quad [\partial_{\overline{z}} f](z) = \sum_{n,m \geqslant 0, n+m \leqslant d} c_{n,m} z^n (m\overline{z}^{m-1}).$$

Thus from part (i),

$$[\Delta f](z) = [\partial_z[\partial_{\overline{z}} f]](z) = \sum_{n,m \geqslant 0, n+m \leqslant d} c_{n,m}(nz^{n-1})(m\overline{z}^{m-1}).$$

Well then f harmonic on $\mathbb{C} \iff \Delta f \equiv 0$ on $\mathbb{C} \iff \sum_{n,m \geqslant 0, n+m \leqslant d} c_{n,m}(nz^{n-1})(m\overline{z}^{m-1}) \equiv 0$ on $\mathbb{C} \iff c_{n,m} = 0$ if n, m are both positive (this last equivalence has forward direction trivial, and for the backwards direction, if $n, m > 0$ and $c_{n,m} \neq 0$, then $c_{n,m}(nz^{n-1})(m\overline{z}^{m-1})$ is non-zero and so the whole polynomial is not identically zero; contradiction). Here we used the fact that such polynomials are identically 0 if and only if all the coefficients are 0; this is **Fact** from the end of Exercise 23(ii).

(iii) If $u : U \to \mathbb{R}$ is a real polynomial $u(x + iy) = \sum_{n,m \geqslant 0, n+m \leqslant d} a_{n,m} x^n y^m$ for $a_{n,m} \in \mathbb{R}$, $d \in \mathbb{N}$, we want to show that u is harmonic $\iff$ it is the real part of a polynomial $f(z) = \sum_{n=0}^{d} c_n z^n$ of one complex variable z. We can rewrite the definition of u by $u(z) = \sum_{n,m \geqslant 0, n+m \leqslant d} a_{n,m}(\frac{z+\overline{z}}{2})^n(\frac{z-\overline{z}}{2i})^m$, which is of the form $\sum_{n,m \geqslant 0, n+m \leqslant d} c_{n,m} z^n \overline{z}^m$. Part (ii) above tells us that u is harmonic on $U \iff c_{n,m} = 0$ if $n, m > 0$, in other words u is of the form $\sum_{0 \leqslant n \leqslant d} c_{n,0} z^n + \sum_{0 \leqslant m \leqslant d} c_{0,m} \overline{z}^m$.

But because u takes purely real values, we have that $u(z) = \overline{u(z)}$, or in other words $\sum_{0\leqslant n\leqslant d}(c_{n,0} - \overline{c_{0,n}})z^n + \sum_{0\leqslant m\leqslant d}(c_{0,m} - \overline{c_{m,0}})\overline{z}^m \equiv 0 \iff c_{n,0} = \overline{c_{0,n}}$ and $c_{0,m} = \overline{c_{m,0}}$ for all $0 \leqslant n, m \leqslant d$ (again using the **Fact** I referenced in part (ii) above). Plugging this information back in we have $u(z)$ is of the form $\sum_{0\leqslant n\leqslant d} c_{n,0}z^n + \sum_{0\leqslant n\leqslant d} \overline{c_{n,0}z^n} = \sum_{0\leqslant n\leqslant d} 2\,\mathfrak{Re}(c_{n,0}z^n)$ (using the formula $2\,\mathfrak{Re}(w) = w + \overline{w}$), and since complex addition is done component wise (real parts together, and imaginary parts together), this is precisely $\mathfrak{Re}(\sum_{0\leqslant n\leqslant d} 2c_{n,0}z^n)$, and we are done.

Exercise S&S §1.13

Suppose that $f = \mathfrak{Re}(f) + i\,\mathfrak{Im}(f) =: u + iv$ is holomorphic (i.e. complex differentiable everywhere) in an open set $\Omega \subseteq \mathbb{C}$. Recall the Cauchy-Riemann equations from class, where $f = u + iv$ being complex differentiable at a point z_0 implies that $\frac{\partial u}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0)$ and $\frac{\partial v}{\partial x}(z_0) = -\frac{\partial u}{\partial y}(z_0)$. **Be aware** that I abbreviate $\frac{\partial u}{\partial x}$ by $\partial_x u$ sometimes. We prove that f is constant in any one of the following cases:

(a) $\underline{\mathfrak{Re}(f) \text{ is constant}}$: since f is holomorphic i.e. complex differentiable everywhere on Ω, we have that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ on all of Ω. $\mathfrak{Re}(f) =: u$ being constant implies that all these partial derivatives are 0 everywhere on Ω (since u constant $\implies \partial_x u = \partial_y u = 0$ on all of Ω).

Then, v must be constant too, since if it were not, there would be $v(x_0 + iy_0) \neq v(x_1 + iy_1)$, implying that either $v(x_0 + iy_0) \neq v(x_1 + iy_0)$ or $v(x_1 + iy_0) \neq v(x_1 + iy_1)$. We take the first case w.l.o.g. (second case we just look at the y-partial instead). Since v takes values in $\mathbb{R}$, the mean value theorem applies (only requires differentiability everywhere), telling us that there then must be some $x^* \in (x_0, x_1)$ s.t. $[\partial_x v](x^* + iy_0) = \frac{v(x_1+iy_0)-v(x_0+iy_0)}{x_1-x_0} \neq 0$, contradicting that $\partial_x v = 0$ on Ω.

Finally, $u = \mathfrak{Re}(f)$ and $v = \mathfrak{Im}(f)$ being constant imply that $f = u + iv$ is constant on Ω.

(b) $\underline{\mathfrak{Im}(f) \text{ is constant}}$: defining $g = -if$, we have that $\mathfrak{Re}(g) = \mathfrak{Im}(f)$ and of course g is holomorphic on Ω, and so by part (a), g is constant on Ω. Multiplying again by i, we get that $f = ig$ is constant on Ω as well.

(c) $\underline{|f| \text{ is constant}}$: if $f = 0$ anywhere on Ω, then $|f| = 0$ there too, but because $|f|$ is constant, $|f| \equiv 0$ identically on Ω, so $f \equiv 0$ identically on Ω as well and is constant. So now suppose f is never zero on Ω. Then, $\frac{1}{f}$ is holomorphic on Ω (since f is), and so $\overline{f} = |f|^2\frac{1}{f}$ is holomorphic on Ω as well. Furthermore, $\mathfrak{Re}(f) = \frac{f+\overline{f}}{2}$ and $\mathfrak{Im}(f) = \frac{f-\overline{f}}{2}$, and sums and scalar multiples of holomorphic functions are holomorphic, so $u = \mathfrak{Re}(f), v = \mathfrak{Im}(f)$ are holomorphic on Ω. But $\mathfrak{Im}(u) = 0$ everywhere on Ω (since $u = \mathfrak{Re}(f)$ only takes values in $\mathbb{R}$), so by part (b) holomorphic u with constant imaginary part must be constant. So now $u = \mathfrak{Re}(f)$ is constant, so by part (a) f must be constant, as desired.

246A Homework 1

Daniel Rui - 10/8/21 to 10/10/21

The problems titled with "Exercise" come from Terry Tao's Notes 0, and the problem titled with "Exercise S&S" come from Stein & Shakarchi (the number after the § symbol is the chapter number, and the number after the period is the exercise number).

Exercise 4

(Uniqueness up to isomorphism of the complex field) Suppose that we are given two complex fields $\mathbb{C} = (\mathbb{C}, i)$ and $\mathbb{C}' = (\mathbb{C}', i')$ (using Definition 1 in Notes 0, which defines a *complex field* to be a pair $(\mathbb{C}, i)$ where i satisfies $i^2 + 1 = 0$, and $\mathbb{C} = \mathbb{R}(i)$ in the language of field extensions). Then, these two complex fields are isomorphic in the sense that there is a unique field isomorphism $\varphi : \mathbb{C} \to \mathbb{C}'$ that maps $i \mapsto i'$ and is the identity on $\mathbb{R}$. The "unique" in the "unique field isomorphism" comes from the fact that if we have two field isomorphisms $\varphi, \varphi' : \mathbb{C} \to \mathbb{C}'$ that satisfy $[i \mapsto i', \varphi|_{\mathbb{R}} = \varphi'|_{\mathbb{R}} = \mathrm{id}_{\mathbb{R}}]$, then I claim that $F := \{z \in \mathbb{C} : \varphi(z) = \varphi'(z)\}$ is a field (subfield of $\mathbb{C}$), which obviously contains both $\mathbb{R}$ and i:

- Associativity/Commutativity/Distributativity: the operations are inherited form $\mathbb{C}$ (since $F \subseteq \mathbb{C}$), and because addition and multiplication in $\mathbb{C}$ were associative and commutative and satisfied the distributative law, they are in F as well.
- Closedness: if $z, w \in \mathbb{C}$ are s.t. $\varphi(z) = \varphi'(z)$ and $\varphi(w) = \varphi'(w)$, then $\varphi(z + w) = \varphi(z) + \varphi(w) = \varphi'(z) + \varphi'(w) = \varphi'(z + w) \in F$ and similarly for multiplication.
- Identities: $0, 1$ are respectively the additive and multiplicative identities, which are in $\mathbb{R} \subseteq F$
- Inverses: if non-zero $z \in F \subseteq \mathbb{C} \implies \varphi(z) = \varphi'(z) \implies \varphi(z)^{-1} = \varphi'(z)^{-1}$, then $z^{-1} \in \mathbb{C}$ satisfies $\varphi(z^{-1}) = \varphi(z)^{-1} = \varphi'(z)^{-1} = \varphi'(z^{-1})$, so $z^{-1} \in F$.

Thus, because $\mathbb{C} := \mathbb{R}(i)$ which is the smallest field/intersection of all fields containing $\mathbb{R}$ and i, and we just proved that F is such a field $\subseteq \mathbb{C}$, we must have $\mathbb{C} = F$, so indeed $\varphi = \varphi'$, and similarly for additive inverses.

Now we just have to prove that there exists a field isomorphism $\varphi : \mathbb{C} \to \mathbb{C}'$ satisfying $[i \mapsto i', \varphi|_{\mathbb{R}} = \mathrm{id}_{\mathbb{R}}]$. Recall from class or Notes 0 that we proved that $(\mathbb{R}[x]/\langle x^2 + 1\rangle, x + \langle x^2 + 1\rangle)$ is a complex field. Let's abbreviate it $(\mathbb{A}, [x])$ where the $\mathbb{A}$ stands for "**a**lgebraically constructed complex field". We do know that $\mathbb{A} = \{a + b[x] : a, b \in \mathbb{R}\}$. Now consider $\sigma : \mathbb{A} \to \mathbb{C}$ defined by $a + b[x] \mapsto a + bi$, where $a + bi$ clearly is an element of $\mathbb{R}(i) =: \mathbb{C}$. Observe that it is a field homomorphism, because $\sigma((a+b[x])+(c+d[x])) = \sigma((a + c) + (b + d)[x]) = (a + c) + i(b + d) = (a + bi) + (c + di) = \sigma(a + b[x]) + \sigma(c + d[x])$ and $\sigma((a + b[x])(c + d[x])) = \sigma((ac - bd) + (ad + bc)[x]) = (ac - bd) + i(ad + bc) = (a + bi)(c + di) = \sigma(a + b[x])\sigma(c + d[x])$.

Then, because field homomorphisms map subfields to subfields, we have that the image $\sigma(\mathbb{A})$ is a subfield of $\mathbb{C}$. However, $\mathbb{R}$ and i are clearly contained in $\sigma(\mathbb{A})$, so again because $\mathbb{C} := \mathbb{R}(i)$ which is the smallest field/intersection of all fields containing $\mathbb{R}$ and i, and we just proved that $\sigma(\mathbb{A})$ is such a field $\subseteq \mathbb{C}$, we must have $\mathbb{C} = \sigma(\mathbb{A})$. In other words, we have shown that σ is surjective, and so indeed σ is a field isomorphism. Finally, by symmetry the same is true for $\tau : \mathbb{A} \to \mathbb{C}'$, and because a composition of field isomorphisms is an isomorphism, we see that $\tau \circ \sigma^{-1} : \mathbb{C} \to \mathbb{C}'$ is our desired field isomorphism.

Exercise 11

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be an isometry of the Euclidean plane that fixes the origin $(0, 0)$ (i.e. it preserves all distances). Then, it must be that T is either a rotation around the origin by some angle $\theta \in \mathbb{R}$, or a reflection across some line passing through the origin. We are provided a hint: by composing T with rotations and/or reflections to transform T into the identity, and by then composing by the inverses of those rotations and/or reflections, we get that T is itself a composition of rotations and/or reflections.

Geometric Perspective

Notation in this section explained more thoroughly in "Linear Algebra Perspective" below since that was written first. So we're given an isometry $T : \mathbb{R}^2 \to \mathbb{R}^2$, that say maps $\mathbf{e}_1 := (1, 0) \mapsto \alpha$. By the definition of isometry and because T fixes the origin, we have $|\alpha| = |\alpha - 0| = |\mathbf{e}_1 - 0| = 1 \implies \alpha \in S^1 \subseteq \mathbb{R}^2$, so $\alpha = (\cos\theta, \sin\theta)$ for some angle θ. Thus, using our intuitive/non-rigorous understanding of the words "rotation" and "angle", there is rotation $\rho_{-\theta}$ by an angle of $-\theta$ that maps $\alpha \mapsto \mathbf{e}_1$, is an isometry, and has an inverse rotation by an angle of θ (denoted ρ_θ). Hence, $\rho_{-\theta} \circ T$ is an isometry that fixes 0 and $\mathbf{e}_1$.

Because $\mathbf{e}_2 = (0, 1)$ is distance 1 from 0 and distance $\sqrt{2}$ from $\mathbf{e}_1$, we must have $[\rho_{-\theta} \circ T](\mathbf{e}_2)$ is also distance 1 from 0 and distance $\sqrt{2}$ from $\mathbf{e}_1$, i.e. $[\rho_{-\theta} \circ T](\mathbf{e}_2)$ must be one of the two points of intersection between the circles $\partial B(0, 1)$ and $\partial B(\mathbf{e}_1, \sqrt{2})$, i.e. the two points $\{\mathbf{e}_2, -\mathbf{e}_2\}$. Defining $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ to map $\mathbf{e}_1 \mapsto \mathbf{e}_1, \mathbf{e}_2 \mapsto -\mathbf{e}_2$ (note that $\tau^2 = \mathrm{id} \iff \tau = \tau^{-1}$, and τ is an isometry), we see that $\rho_{-\theta} \circ T = \tau^{0,1}$, implying that $T = \rho_\theta \circ \tau^{0,1}$. Again, using our intuitive understanding of the words "rotation" and "reflection across a line", $T = \rho_\theta \circ \tau^0 = \rho_\theta$ is precisely a rotation by an angle θ, and $T = \rho_\theta \circ \tau^1$ is a reflection across the x-axis that is rotated by an angle of θ, i.e. is a reflection across a line passing through the origin tilted at an angle of θ.

Corollary (i.e. part (ii) of this exercise): looking back at Notes 0, we see that using a continuity argument for $\det(\bullet)$ on the connected topological space S^1 (and $\det([z \mapsto 1z]) = 1$) we get that $\mathscr{A} := \{[z \mapsto \omega z] : \mathbb{C} \to \mathbb{C} : \omega \in S^1\} \subseteq \{\rho \in \mathrm{GL}(2, \mathbb{R}) : \det(\rho) = 1, \rho \text{ isometry}\} =: \mathscr{C}$. The same continuity argument applied to angles $\theta \in \mathbb{R}$ gives that $\mathscr{B} :=$\{intutive set of of rotations ρ_θ by an angle θ\} $\subseteq \mathscr{C}$. Above we showed that any isometry $T \in \mathrm{GL}(2, \mathbb{R})$ can be written as $T = \rho_\theta \circ \tau^{0,1}$, and if we further assume $\det(T) = 1$, then we get in fact $T = \rho_\theta$ (superscript of τ can not possibly be 1 because then applying determinant to both sides and using multiplicativity, we get $1 = -1$; contradiction), implying that $\mathscr{B} = \mathscr{C}$. Finally, $\mathscr{B} \subseteq \mathscr{A}$ because for any angle θ, the rotation $\rho_\theta \in \mathscr{B}$ corresponds to $f_\theta := [z \mapsto (\cos\theta + i \sin\theta)z] \in \mathscr{A}$, because f_θ and ρ_θ are both in $\mathscr{C}$ and map $1 \mapsto (\cos\theta + i \sin\theta)$, and there is only one map in $\mathscr{C}$ that does that. So $\mathscr{A} = \mathscr{B} = \mathscr{C}$.

Thus, all ρ_θ correspond to $[z \mapsto (\cos\theta + i\sin\theta)z]$ and τ of course corresponds to complex conjugation $[z \mapsto \overline{z}]$, so since we just proved that any isometry $T : \mathbb{C} \to \mathbb{C}$ (identifying $\mathbb{C} \simeq \mathbb{R}^2$) that fixes 0 is of the form $T = \rho_\theta \circ \tau^{0,1}$, we have that any such isometry of $\mathbb{C}$ equals ωz or $\omega\overline{z}$ where $\omega = (\cos\theta + i\sin\theta) \in S^1$. Finally, for arbitrary isometries T of $\mathbb{C}$, 0 maps to say $z_0 \in \mathbb{C}$, meaning that $[T - z_0]$ is an isometry fixing 0, so $[T - z_0] = \omega z$ or $\omega\overline{z} \iff T = z_0 + \omega z$ or $z_0 + \omega\overline{z}$. ■

Linear Algebra Perspective

First, using the interpretation/definition of the determinant as the *signed volume of a lineary transformation*, we have that $\det(T) = \pm 1$, i.e. areas don't change under isometries (one could see this by noting that the unit square is mapped to another unit square, since the side lengths remain 1 and the diagonals remain $\sqrt{2}$; or from another point of view areas/volumes are just integrals of lengths, so if the lengths are unchanged, so are the areas/volumes).

Defining $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ (notation chosen for its "reflection symmetry") to map $(1,0) \mapsto (1,0)$ and $(0,1) \mapsto (0,-1) = -(0,1)$, the multilinearity of the determinant gives $\det(\tau) = -1$. Note that $\tau^2 = \text{id} \iff \tau = \tau^{-1}$, and τ is an isometry (because τ maps $(a,b) \mapsto (a,-b)$ and the Pythagorean theorem tells us both lengths are $\sqrt{a^2+b^2}$). By the multiplicativity of the determinant, if $\det(T) = -1$, then $\det(\tau \circ T) = (-1)(-1) = 1$. Thus, we can always write $\tau^{0,1} \circ T = \rho \iff T = \tau^{0,1} \circ \rho$ for a map ρ with determinant 1 (notation chosen for the "circular shape" and the sound "rho"(-tation)) and where the superscript of τ is 0 if $\det(T) = 1$ and 1 if $\det(T) = -1$. Because τ, T are both isometries and $\tau^{0,1} \circ T = \rho$, ρ must also be an isometry.

Assuming/defining that the set of "rotations" is $= \{\rho \in \mathrm{GL}(2,\mathbb{R}) : \det(\rho) = 1, \rho \text{ isometry}\}$ and the set of "reflections" ("flips") is $= \{f \in \mathrm{GL}(2,\mathbb{R}) : \det(f) = -1, f \text{ isometry}\}$, we do indeed get that T must in fact lie in one of these two sets, as desired.

Exercise 14

In this problem, we examine a variety of subgroups of $\mathbb{C}$.

(i) Recall the following facts about the exponential and trigonometric functions: for the complex exponential e^z (which we defined as the power series $\sum_{k=0}^{\infty} \frac{z^k}{k!}$ for all $z \in \mathbb{C}$), $e^{x+y} = e^x e^y$ (so in particular for $n \in \mathbb{N}$, $(e^x)^n = e^{nx}$), $e^{2\pi i k} = 1$ for all $k \in \mathbb{Z}$, and 2π is the smallest $t \in \mathbb{R}^+$ s.t. $e^{ti} = 1$ (or equivalently 2π is the smallest positive real number x s.t. $\cos(x) = 1$). Then, $e^{2\pi i/n}$ is a complex nth root of unity (since $(e^{2\pi i/n})^n = e^{2\pi i} = 1$), as are all the elements of $\{e^{2\pi i k/n}\}_{i=1}^{n}$ (since $(e^{2\pi i k/n})^n = e^{2\pi i k} = 1$).

All these elements are distinct, because if not, we would have $e^{2\pi i k_1/n} = e^{2\pi i k_2/n} \neq 0$ (w.l.o.g. $k_1 < k_2$, $k_1, k_2 \in \{1, \ldots, n\}$) so dividing yields $e^{2\pi i(k_2-k_1)/n} = 1$, contradicting the fact I stated earlier that "2π is the smallest $t \in \mathbb{R}^+$ s.t. $e^{ti} = 1$". As $x^n - 1 = 0$ has $\leqslant n$ roots in $\mathbb{C}$ (using the division algorithm one can prove that over any integral domain R, such as the field $\mathbb{C}$, $R[x]$

is also an integral domain and so any degree n polynomial has $\leqslant n$ roots) and we have already found n distinct roots, this set is all the nth complex roots of unity.

Finally, the division algorithm for polynomials tells us that since $\prod_{k=0}^{n-1}(z - e^{2\pi ik/n})$ divides $z^n - 1$, we have $z^n - 1 = q(z) \cdot \prod_{k=0}^{n-1}(z - e^{2\pi ik/n}$, and because the product and $z^n - 1$ are both monic and have the same degree n, we must have that $q(z)$ is identically equal to 1, so indeed $z^n - 1 = \prod_{k=0}^{n-1}(z - e^{2\pi ik/n}$.

Tidbit for part (ii): I claim that $C_n := \{e^{2\pi ik/n}\}_{i=1}^{n} = \langle e^{2\pi i/n}\rangle$. This is quite trivial: we just use that $(e^{2\pi i/n})^k = e^{2\pi ik/n}$, and $e^{2\pi in/n} = e^{2\pi i} = 1$. It is clear that $\langle e^{2\pi i/n}\rangle$ is closed under (complex) multiplication. Recall also that complex multiplication is associative. Finally, the multiplicative identity 1 of $\mathbb{C}$ is in G, and $e^{2\pi i(n-k)/n}$ is the multiplicative inverse of $e^{2\pi ik/n}$, so indeed we have verified all the group axioms and have shown that C_n is a cyclic group.

(ii) We want to show that the only compact subgroups G of $\mathbb{C}^\times$ (the multiplicative group of complex numbers) are the unit circle S^1 and the nth roots of unity C_n (described at the end of part (i) above). First note that if $z \in G$ and $|z| > 1$, then because $z^n \in G$ for all $n \in \mathbb{N}$ where $|z^n| = |z|^n \to \infty$ as $n \to \infty$, we would get that G is unbounded, contradicting compactness (using the Heine-Borel theorem on $\mathbb{R}^2$ which has the same topology as $\mathbb{C}$, compactness $\iff$ closed and bounded). If $z \in G$ with $|z| < 1$, then $|z^{-1}| > 1$, so by the previous sentence that is impossible. Thus, $G \subseteq S^1$. As per the hint, we split into two cases, **Case 1** where 1 is a limit point of G, and **Case 2** where 1 is not a limit point of G.

Case 1: w.l.o.g. we can assume that we have $\{e^{it_n}\}_{n=1}^{\infty} \subseteq G$ for a sequence of positive real numbers $\{t_n\}_{n=1}^{\infty}$ that converge to 0 (otherwise since we're assuming 1 is a limit point of G, there will be a sequence of negative real numbers, so we can just put minus-signs in front of all the t_n in the following argument, and consider $[-2\pi, 0)$ instead). Then, any $t \in (0, 2\pi]$ can be written as $t = \sum_{n=1}^{\infty} a_n t_n$ where $a_n \in \mathbb{Z}_{\geqslant 0}$; essentially choose the a_n "greedily", i.e. the biggest possible a_n s.t. the sum does not exceed t.

Defining $s_n := \sum_{k=1}^{n} a_k t_k$ to be the partial sums, we see that $e^{s_n} = \prod_{k=1}^{n}(e^{it_k})^{a_k}$, and as e^{it_k} are all in G and G is closed under complex multiplication, all the $e^{s_n} \in G$ as well. Using the continuity of e^z (the Weierstrass M-test gives us that on any compact set e^z is the uniform limit of continuous partial sums, so indeed e^z is continuous on all of $\mathbb{C}$), since $s_n \to t$, we get that $e^{s_n} \to e^t$, so by compactness e^t must lie in G. Of course $t \in (0, 2\pi]$ was arbitrary and e^{it} is 2π-periodic, so indeed $S_1 = \{e^{it} : t \in \mathbb{R}\} \subseteq G \subseteq S^1$, so in this case, $G = S^1$.

Case 2: if 1 is not a limit point of G, then the infimum of all positive $t \in \mathbb{R}$ s.t. $e^{it} \in G$ is strictly positive. Call this $t_0 > 0$. Of course since G is a subgroup of $\mathbb{C}^\times$, we must have $1 \in G$. Anyways, there is some smallest $n \in \mathbb{N}$ s.t. $t_0 n \geqslant 2\pi$. Then, we have $2\pi \leqslant t_0 n < 2\pi + t_0$ (if not, i.e. $t_0 n \geqslant 2\pi + t_0$, $t_0(n-1) \geqslant 2\pi$, contradicting the minimality of n). Since $e^{2\pi i} = 1 \in G$ and $e^{it_0} \in G \implies e^{it_0 n} \in G$, we have $\frac{e^{it_0 n}}{e^{2\pi i}} = e^{i(t_0 n - 2\pi)} \in G$. But $t_0 n - 2\pi < t_0$, so in order to not contradict minimality of t_0, we must have $t_0 = \frac{2\pi}{n}$. Then, $C_n \subseteq G$. It can not be that $e^{it_1} \in G$ for any $t_1 \notin t_0\mathbb{R}$, because if it were, we would be able to subtract integer multiples of t_0 from t_1 until we get $e^{i(t_1 - t_0 k)} \in G$ where $t_1 - t_0 k \in [0, t_0)$, contradicting minimality of t_0. Thus, in this case $G = C_n$.

(iii) We are asked to exhibit an example of a non-compact subgroup of S_1. First I claim that $G := \{e^{2\pi i r} : r \in \mathbb{Q}\}$ is a group. We have $1 = e^{2\pi i 0} \in G$, and $r_1, r_2 \in \mathbb{Q} \implies r_1 + r_2 \in \mathbb{Q} \implies e^{2\pi i r_1} e^{2\pi i r_2} = e^{2\pi i (r_1 + r_2)} \in G$, and associativity inherited from associativity of complex multiplication in $\mathbb{C}$, and finally inverses because $e^{2\pi i r} \in G \implies r \in \mathbb{Q} \implies -r \in \mathbb{Q} \implies e^{2\pi i(-r)} \in G$ where $e^{2\pi i r} e^{2\pi i (-r)} = e^{2\pi i 0} = 1 \in G$. Clearly, 1 is a limit point since $\frac{1}{n} \to 0$ as $n \to \infty$ implies $e^{2\pi i (1/n)} \to e^{2\pi i 0} = 1$ (again using continuity of e^z, discussed in part (ii) above), but $G \neq S_1$ because G is countable whereas S^1 is uncountable. Similarly G is countably infinite, not finite like all the C_n, implying that G (a subgroup of S^1) can not be compact (since we classified all compact subgroups of S^1 part (ii) above, and G is not one of them).

(iv) We are allowed to the following facts as black boxes:

- The only closed subgroups of the additive group $\mathbb{R}$ are the whole group $\mathbb{R}$, the trivial group $\{0\}$, and groups of the form αZ for some non-zero $\alpha \in \mathbb{R}$ (depending on whether 0 is isolated or not, and what the smallest positive value of the group is if 0 is isolated).
- The only connected closed subgroups of the additive group $\mathbb{C}$ are the whole group $\mathbb{C}$, the trivial group $\{0\}$, and the lines $z\mathbb{R}$ for some non-zero $z \in \mathbb{C}$ (modify argument of bullet (1), and if subgroup contains line, quotient it out to reduce back to 1-dimensional problem).
- The only closed subgroups of the additive group $\mathbb{C}$ are either $\mathbb{C}$, discrete (a union of isolated points), a line $z\mathbb{R}$ for some non-zero $z \in \mathbb{C}$, or of the form $z\mathbb{R} + w\mathbb{Z}$ where $z, w \in \mathbb{C}$ are non-zero and linearly independent over $\mathbb{R}$ (closed subgroups = unions of connected closed subgroups, and for the last possibility, project onto three subspaces say $\mathbb{R}, i\mathbb{R}, (1+i)\mathbb{R}$, at least two of those projections will yield isolated evenly spaced points, use this data to determine w).

We are asked to show that the only connected closed subgroups G of $\mathbb{C}^\times$ are the whole group $\mathbb{C}^\times$, the trivial group $\{1\}$, and the one-parameter groups of the form $e^{z\mathbb{R}}$ for some $z \in \mathbb{C}^\times$. As the hint suggests, we consider the inverse image of G under the exponential map, and use bullet (3). First because of the multiplicativity of the exponential map, $\exp : (\mathbb{C}, +) \to (\mathbb{C}^\times, \times)$ is a homomorphism, so **IF** G is a connected closed subgroup of $\mathbb{C}^\times$, THEN the inverse image $H := \exp^{-1}(G)$ must be a group in $(\mathbb{C}, +)$. Because $\exp(z)$ is continuous (see discussion in part (ii) above) and we assumed G was closed, H must also be a closed subgroup of $(\mathbb{C}, +)$.

General observation: for any function $f : X \to Y$ and subset $S \subseteq Y$, we have that $f(f^{-1}(S)) = f(\{x \in X : f(x) \in S\} = \{f(x) : x \in X, f(x) \in S\} \subseteq S$. To get equality, it is sufficient (and necessary) to have that the restricting $f : f^{-1}(S) \to S$ is surjective; thus if $f : X \to Y$ is surjective that suffices to show $f(f^{-1}(S)) = S$. Because e^z is surjective (for any $w = r(\cos\theta + i\sin\theta) \in \mathbb{C}$, we have for $z = \ln(r) + i\theta$ that $e^z = e^{\ln(r)} e^{i\theta} = w$), we know now that **IF** G is a connected closed subgroup of $\mathbb{C}^\times$, THEN $H := \exp^{-1}(G)$ is a closed subgroup of $(\mathbb{C}, +)$, and moreoever $G = \exp(H)$. That means that the set $\mathcal{S}$ of connected closed subgroups G of $\mathbb{C}^\times$ is a subset of $\{\exp(H) : H \text{ closed subgroup of } (\mathbb{C}, +)\}$, or in other words, $\mathcal{S} = \{\exp(H) : H \text{ closed subgroup of } (\mathbb{C}, +) \text{ AND } \exp(H) \text{ is a closed connected subgroup of } \mathbb{C}^\times\}$. So by bullet (3), we have four classes of possibilities for H, for which we have to check if $\exp(H)$ is a closed connected subgroup of $\mathbb{C}^\times$:

(a) $\underline{H = \mathbb{C}}$. By surjectivity of $e^z : \mathbb{C} \to \mathbb{C}^\times$, we have $G := \exp(H) = \mathbb{C}^\times$, which is indeed a closed connected subgroup of $\mathbb{C}^\times$ (recall that "closed" here means closed in the subspace topology of $\mathbb{C}^\times$, but the axioms of a topology $(T, \mathcal{T})$ tell us that T is always closed in $\mathcal{T}$; also connected $\iff$ the only sets that are open AND closed (or equiv. have no boundary) are $\varnothing$ and T itself. In this case we can use the fact that $\mathbb{C}^\times$ is path connected, and that path connected $\implies$ connected to see that $G = \mathbb{C}^\times$ is connected).

(b) $\underline{H \text{ is a discrete set}}$. H must be finite/countable (otherwise there will be some compact square $[n, n+1] \times [m, m+1] \subseteq \mathbb{R}^2 \simeq \mathbb{C}$ in which there are infinitely many points, implying we have a limit point by Bolzano-Weierstrass, contradicting discreteness/isolatedness), so $G := \exp(H)$ is also a finite/countable set. Now suppose G was connected; then because connected sets are preserved by continuous functions, projecting G down to the real and imaginary axes via π_1, π_2 yield connected subsets of $\mathbb{R}$ and $i\mathbb{R}$. But we know that the connected subsets of $\mathbb{R} \simeq i\mathbb{R}$ are precisely the intervals, and the only intervals with a finite/countable number of points are precisely the singletons. And the only singleton that is also a subgroup of $\mathbb{C}^\times$ is precisely $\{1\}$, which indeed is closed in $\mathbb{C}^\times$, so in this class of H the only $\exp(H) \in \mathcal{S}$ is precisely $\{1\}$.

(c) $\underline{H = z\mathbb{R},\ 0 \neq z \in \mathbb{C}}$. Then $G := \exp(z\mathbb{R}) = \{e^{ra+rbi} : r \in \mathbb{R}\}$ where $z = a + bi$, $a, b \in \mathbb{R}$. We check easily that G is a group: the identity is $e^{0(a+bi)} = 1$, associativity (and commutativity) is inherited, $e^{r_1(a+bi)}e^{r_2(a+bi)}e^{(r_1+r_2)(a+bi)}$, and $e^{r(a+bi)}e^{(-r)(a+bi)} = 1$. It's connected because $\exp(z)$ is a continuous function and hence maps connected sets (like $z\mathbb{R}$, which is path-connected) to connected sets. Closedness is a bit trickier, since it's not true that continuous images of closed sets are closed. But continuous images of compact sets are compact, hence closed (by Heine-Borel)! So consider $z[n, n+1] \subseteq z\mathbb{R}$ for $n \in \mathbb{Z}$, which is a compact subset of $\mathbb{C}$. If $z \in i\mathbb{R} \iff a = 0$, then $\exp(z\mathbb{R}) = S^1$, which we already know to be a subgroup of $\mathbb{C}^\times$, and is obviously connected (it's path-connected) and closed. So we now assume $a < 0$ or $a > 0$.

Note $w \in \exp(z[n, n+1]) \implies |w| \in [e^{an}, e^{a(n+1)}] \implies \exp(z[n, n+1]) \subseteq \overline{A(e^{an}, e^{a(n+1)})} =:$ the closed annulus with inner/outer radii the 1st/2nd components respectively. We'll abbreviate this closed annulus as A_n. Note that $\bigcup_{n \in \mathbb{Z}} A_n = \mathbb{C}^\times$, and that for even (similarly odd) n, m, the annuli A_n, A_m have positive distance from each other. Since $z[n, n+1]$ is compact, $\exp(z[n, n+1])$ is closed (as discussed in previous paragraph). I claim that $B_0 := \bigcup_{n \text{ even}} \exp(z[n, n+1])$ is closed in $\mathbb{C}^\times$ (similarly for n odd), because for any convergent sequence in B_0 that converges to $w_0 \in \mathbb{C}^\times$, there must be some even $n \in \mathbb{Z}$ s.t. $w_0 \in A_n$ (if there wasn't such an n, then that would mean w_0 is in the interior of some A_n for n odd, so it would have positive distance from all the A_n for n even, meaning there is no way for a convergent sequence in B_0 to converge to it; contradiction), and I claim that the tail of the convergent sequence in B_0 must lie completely within A_n.

This is because if not, we would have infinitely many elements of the sequence in other A_m for even $m \neq n$, but because other A_m for even m have positive distance from A_n, there would be no way for the sequence to converge. Thus, the tail end of the sequence in B_0 lies completely in the closed $\exp(z[n, n+1])$, so its limit w_0 lies in $\exp(z[n, n+1]) \subseteq B_0$ as well. A similar argument proves that $B_1 := \bigcup_{n \text{ odd}} \exp(z[n, n+1])$ is also closed. Thus, $\exp(z\mathbb{R}) = \bigcup_{n\in\mathbb{Z}} \exp(z[n, n+1]) = B_0 \cup B_1$ is a finite union of closed sets, and is hence closed. Thus in this class of H, all the $\exp(z\mathbb{R})$ for non-zero $z \in \mathbb{C}$ are in $\mathcal{S}$.

(d) $\underline{H = z\mathbb{R} + w\mathbb{Z},\ 0 \neq z, w \in \mathbb{C}}$. If $z \in i\mathbb{R}$ then we can write $H = i\mathbb{R} + \alpha\mathbb{Z}$ for $\alpha \in \mathbb{R}$, and if $z \notin i\mathbb{R}$, then we can write $H = z\mathbb{R} + \alpha i\mathbb{Z}$ for $\alpha \in \mathbb{R}$. In the first case, $\exp(i\mathbb{R} + \alpha\mathbb{Z}) = \exp(i\mathbb{R})\exp(\alpha\mathbb{Z}) = S^1 \exp(\alpha\mathbb{Z}) = \bigcup_{n\in\mathbb{Z}} e^{\alpha n} S^1$, which is not connected for $\alpha \neq 0$ (each $e^{\alpha n} S^1$ is a scaled copy of S^1 with positive distance from any other $e^{\alpha m} S^1$), and is simply S^1 for $\alpha = 0$ which we already found to be in $\mathcal{S}$ in part (c) above.

In the other case ($z = a + bi \notin i\mathbb{R} \iff a \neq 0$), we have $\exp(z\mathbb{R} + \alpha i\mathbb{Z}) = \exp(z\mathbb{R})\exp(\alpha i\mathbb{Z}) = \bigcup_{n\in\mathbb{Z}} e^{i\alpha n} \exp(z\mathbb{R})$. If α is an integer multiple of 2π, then this is precisely $\exp(z\mathbb{R})$, which we know to be in $\mathcal{S}$ already by part (c) above. Else if α is a rational multiple of 2π, say $2\pi\frac{k}{n}$ where $\frac{k}{n}$ is in lowest terms (equivalent to taking $\alpha = 2\pi\frac{1}{n} = \frac{2\pi}{n}$, related to the fact that for k relatively prime to n, $[k]$ generates the group $(\mathbb{Z}/n\mathbb{Z}, +)$), then this union will be a finite union of n rotated copies of the spiral $e(z\mathbb{R})$, which is not connected (EDIT 11/15/21:) since $U_1 := \exp(i(-\frac{1}{3n}, \frac{1}{3n}))\exp(z\mathbb{R}) = \exp(i(-\frac{1}{3n}, \frac{1}{3n}) + z\mathbb{R})$ and $U_2 := \exp(i(\frac{2}{3n}, 2\pi - \frac{2}{3n}))\exp(z\mathbb{R})$ are disjoint open sets (by say, the open mapping theorem) that cover the n rotated copies of the spiral.

Else, α is not a rational multiple of 2π, so $\bigcup_{n\in\mathbb{Z}} e^{i\alpha n}\exp(z\mathbb{R})$ would be an infinite union of rotated copies of the spiral, whose closure (in $\mathbb{C}^\times$) I **claim** is $\mathbb{C}^\times$ itself. Let rS^1 be some circle, so $r \in \mathbb{R}^+$. Then because $e^{at} : \mathbb{R} \to \mathbb{R}+$ is bijective, for any fixed $n \in \mathbb{N}$ there is exactly one point at which $e^{i\alpha n}\exp(z\mathbb{R})$ hits rS^1, namely $e^{i\alpha n} e^{(t(a+bi)} \in rS^1$ for $t := \frac{\ln(r)}{a}$. So the intersection of this union and the circle rS^1 is $\{r\exp(i(\alpha n + tb)\}_{n\in\mathbb{Z}}$, i.e. a rotation of the $\{r\exp(i\alpha n)\}_{n\in\mathbb{Z}}$ by e^{itb} where $t, b \in \mathbb{R}$ are just constants. To prove the **claim** stated at the beginning of this paragraph, it suffices to show that $\{r\exp(i\alpha n)\}_{n\in\mathbb{Z}}$ is dense in rS^1, or equivalently $\{\exp(i\alpha n)\}_{n\in\mathbb{Z}}$ is dense in S^1.

Once we do that (we do it in the "Appendix" subsection to this problem, see below), then we see that because $\{r\exp(i\alpha n)\}_{n\in\mathbb{Z}}$ is countable and rS^1 is uncountable, the union $\exp(H) = \bigcup_{n\in\mathbb{Z}} e^{i\alpha n}\exp(z\mathbb{R})$ can not equal $\mathbb{C}^\times$, i.e. equal its closure, meaning it's not closed, and hence not in $\mathcal{S}$. As this point we will be done with the entire problem, since that means that for this class of H, the only $\exp(H) \in \mathcal{S}$ are the ones we already accounted for. Therefore, indeed we will have shown that $\mathcal{S} = \{\mathbb{C}^\times, \{1\}\} \cup \bigcup_{0\neq z\in\mathbb{C}} \exp(z\mathbb{R})$, as desired.

Appendix

Claim: for α not a rational multiple of 2π, $\{\exp(i\alpha n)\}_{n\in\mathbb{Z}}$ is dense in S^1. *Proof:* let $a = e^{2\pi i\alpha}$ with $\alpha \notin \mathbb{Q}$. We want to show that the the sequence $\{a_n\}_{n=1}^{\infty}$ ($a_n := a^n$) is dense in the unit circle $\mathbb{T} := \{|z| = 1\}$ (obviously the sequence is contained within $\mathbb{T}$).

Well, first recall that $e^{i\theta}$ is a 2π-periodic function ($\theta \in \mathbb{R}$) that moreover doesn't repeat before the period is over (i.e. for any $\theta_0 \in \mathbb{R}$, $e^{i\theta} : [\theta_0, \theta_0 + 2\pi) \to \mathbb{C}$ is an injective function, because $e^{i\theta}$ just walks around the unit circle which does not repeat points until the next period starts, i.e. when the point $e^{i\theta_0}$ is reached again at $\theta = \theta_0 + 2\pi$). Let us call this "doesn't repeat before the period over" fact the "non-repetition property".

Now we consider the sequence $\{a_n\}_{n=1}^{\infty}$. I claim that the sequence can never repeat: if it did, say $a^n = a^m$ for some $n, m \in \mathbb{N}$ where $n \neq m$, then by 2π-periodicity and the non-repetition property, it must be that $2\pi\alpha n \equiv 2\pi\alpha m \pmod{2\pi}$, i.e. there is $k \in \mathbb{Z}$ s.t. $2\pi\alpha(n - m) = 2\pi k \iff \alpha(n - m) = k$. But $n, m, k \in \mathbb{Z}$ and $n \neq m$, implying that $\alpha = \frac{k}{n-m} \in \mathbb{Q}$; contradiction.

To prove that the sequence is dense in $\mathbb{T}$, it suffices to show that for any fixed $z_0 \in \mathbb{T}$ and $\epsilon > 0$, there is always some $a^n \in \{a_n\}_{n=1}^{\infty}$ s.t. $|a^n - z_0| < \epsilon$. Choose some $N \in \mathbb{N}$ s.t. $N < 2\pi\frac{1}{\epsilon}$ (e.g. $N = \lceil 2\pi\frac{1}{\epsilon}\rceil + 1$), and let us divide $\mathbb{T} = \{e^{2\pi i\theta} : \theta \in [0, 1)\}$ (equality holds because of 2π-periodicity of $e^{i\theta}$) into N arcs $A_1, \ldots, A_N$ say by letting $A_i := \{e^{2\pi i\theta} : \theta \in [\frac{i-1}{N}, \frac{i}{N})\}$. These arcs are all disjoint from each other by the non-repetition property.

Because the sequence never repeats, the pigeonhole principle guarantees that among $\{a_n\}_{n=1}^{N+1}$, there must be some A_i containing at least two $a^n \in \{a_n\}_{n=1}^{N+1}$, say a^m and a^n where $m < n$ and of course $m, n \in [N + 1]$. Let $\eta \in [-\pi, \pi)$ be the angle (in radians) between a^m and a^n; that is to say, $\eta \equiv 2\pi\alpha(n - m) \pmod{2\pi}$ s.t. $\eta \in [-\pi, \pi)$ (in fact, $|\eta| < \frac{2\pi}{N}$ because a^n and a^m are in the same A_i) and $a^m e^{i\eta} = a^n$. Since $a^n \neq a^m$, $|\eta| > 0$. Let us now suppose w.l.o.g. that $\eta > 0$ (if it were negative, we would just go clockwise around the unit circle instead, i.e. going from A_i to A_{i-1}, etc). **We prove that** given $a^k \in A_i$, there is some amount of steps we can take s s.t. $a^k e^{s\eta} = a^{k+s(n-m)}$ s.t. $a^{k+s(n-m)} \in A_{i+1}$ (where indices of the A_i are taken modulo N).

Let us define $\theta_0 \in [0, 2\pi)$ s.t. $\theta_0 \equiv 2\pi\alpha k \pmod{2\pi}$ (i.e. the argument of a^k expressed as a number in $[0, 2\pi)$). Then, $a^k \in A_i$ means that $\theta_0 \in [2\pi\frac{i-1}{N}, 2\pi\frac{i}{N})$. Because $\eta > 0$, the sequence $\{\theta_0 + t\eta\}_{t=0}^{\infty}$ is increasing, and so it will eventually exceed $2\pi\frac{i}{N}$. Letting s be the first $t \in \mathbb{N}$ s.t. $\theta_0 + t\eta \geqslant 2\pi\frac{i}{N}$, I claim that $e^{\theta_0 + s\eta} = a^{k+s(n-m)} \in A_{i+1}$. This is because $\eta < \frac{2\pi}{N}$, so $\theta_0 + (s - 1)\eta < 2\pi\frac{i}{N} \implies \theta_0 + s\eta < 2\pi\frac{i+1}{N}$. By definition of s, $\theta_0 + s\eta \geqslant 2\pi\frac{i}{N}$, and so indeed $\theta_0 + s\eta \in [2\pi\frac{i}{N}, 2\pi\frac{i+1}{N}) \iff e^{\theta_0 + s\eta} = a^{k+s(n-m)} \in A_{i+1}$, as desired.

Clearly, this argument shows that for any arc A_i, there is some $a^k \in \{a^n\}_{n=1}^{\infty}$ s.t. $a_k \in A_i$ (given $a^1 \in A_{i_0}$, we know that $a^{1+s(n-m)} \in A_{i_0+1}, a^{1+2s(n-m)} \in A_{i_0+2}$, and so on. Eventually we'll get to A_i). Letting A be the arc containing z_0 (fixed above), we know that there is some $k \in \mathbb{N}$ s.t. $a^k \in A$.

The arclength along the circle between a^k and z_0 is less than the arclength of A, which is $\frac{2\pi}{N}$, which recall is $< \epsilon$. But the shortest path between a^k and z_0 is a line, where the length of this line is $|a^k - z_0|$, implying that $|a^k - z_0| < \epsilon$, as desired.

Exercise 16

Let $\{z_n\}_{n=1}^{\infty} \subseteq \mathbb{C}$. **Claim:** $\sin(z_n)$ is bounded $\iff$ $\mathfrak{Im}(z_n)$ is bounded, and similarly with $\sin(z_n)$ replaced by $\cos(z_n)$. First recall from class that we defined $\sin(z), \cos(z), \exp(z)$ using the Taylor series on $\mathbb{R}$ and plugging in complex values instead, from which we derived the formulas $\sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$ and $\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$ for all $z \in \mathbb{C}$. Now for the proof of the claim.

Proof: ($\Longleftarrow$) suppose there is some $B \in \mathbb{N}$ s.t. $|y_n| \leqslant B$ for all $n \in \mathbb{N}$ where $x_n + iy_n =: z_n$ and $x_n, y_n \in \mathbb{R}$. Then (last step we use monotonicity of e^x on $\mathbb{R}$),

$$|\sin(z_n)| \leqslant \frac{|e^{ix_n}e^{-y_n}| + |e^{-ix_n}e^{y_n}|}{|2i|} = \frac{|e^{-y_n}| + |e^{y_n}|}{2} \leqslant \frac{e^B + e^B}{2} = e^B,$$

so indeed $|\sin(z_n)|$ is bounded. Indeed doing the same thing to $\cos(z_n)$, we also get the same bound $|\cos(z_n)| \leqslant \frac{|e^{-y_n}|+|e^{y_n}|}{2} \leqslant \frac{e^B+e^B}{2} = e^B$.

($\Longrightarrow$): using the reverse triangle inequality we have

$$|\sin(z_n)| \geqslant \frac{\big||e^{ix_n}e^{-y_n}| - |e^{-ix_n}e^{y_n}|\big|}{|2i|} = \frac{|e^{-y_n} - e^{y_n}|}{2} = \frac{e^{|y_n|} - e^{-|y_n|}}{2} \geqslant \frac{e^{|y_n|} - 1}{2},$$

and similarly $|\cos(z_n)| \geqslant \frac{\big||e^{-y_n}|-|e^{y_n}|\big|}{2} \geqslant \frac{e^{|y_n|}-1}{2}$, so if we have that all $|\sin(z_n)|$ or $|\cos(z_n)|$ are $\leqslant B$ for some bound $B \in \mathbb{N}$, then the above inequalities show that $e^{|y_n|} \leqslant 2B+1 \implies |y_n| \leqslant \ln(2B+1)$ (where $\ln(x)$ we know from real analysis is defined on $(0,\infty)$ and satisfies $\ln(e^x) = x$ for all $x \in \mathbb{R}$), so indeed we have that the imaginary parts of z_n must be bounded. ■

Exercise 18

Given the definition that $\exp(z) := \sum_{k=0}^{\infty} \frac{z^k}{k!}$ for any $z \in \mathbb{C}$, we want to show $\exp(z) = \lim_{n\to\infty}(1+\frac{z}{n})^n$. No cleverness/tricks here, let's dive right in. Using the binomial theorem, we get that for any $n \in \mathbb{N}, z \in \mathbb{C}$, $(1+\frac{z}{n})^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!\,k!}(\frac{z}{n})^k$, so

$$\begin{aligned}
\left|\left(1+\frac{z}{n}\right)^n - \exp(z)\right| &= \left|\sum_{k=0}^{n} \frac{n!}{(n-k)!\,n^k}\frac{z^k}{k!} - \sum_{k=0}^{\infty}\frac{z^k}{k!}\right| \\
&= \left|\sum_{k=0}^{n}\left(\frac{n!}{(n-k)!\,n^k} - 1\right)\frac{z^k}{k!} - \sum_{k=n+1}^{\infty}\frac{z^k}{k!}\right| \\
&\leqslant \sum_{k=0}^{n}\left|\frac{n(n-1)\cdots(n-(k-1))}{n^k} - 1\right| \cdot \frac{|z|^k}{k!} + \sum_{k=n+1}^{\infty}\frac{|z|^k}{k!}.
\end{aligned}$$

Let us now fix $\epsilon > 0$, $z \in \mathbb{C}$, and define $R := |z| < \infty$. Because the series $\exp(x) := \sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges absolutely for all $x \in \mathbb{R}$, there is some N large enough s.t. $n \geqslant N \implies \sum_{k=n+1}^{\infty} \frac{|z|^k}{k!} < \frac{\epsilon}{2}$. Also observe that for any $k \in \mathbb{N}$, $\frac{R^k}{k!} \leqslant \sum_{k=0}^{\infty} \frac{R^k}{k!} = \exp(R)$. Then we have for $n \geqslant N$ and our z fixed:

$$\left|\left(1+\frac{z}{n}\right)^n - \exp(z)\right| \leqslant \exp(R) \cdot \sum_{k=0}^{N} \left|\frac{n(n-1)\cdots(n-(k-1))}{n^k} - 1\right| + \frac{\epsilon}{2}.$$

For any $k \in \mathbb{N}$, we have that as $n \to \infty$, the function $\frac{n!}{(n-k)!\, n^k} = \frac{n(n-1)\cdots(n-(k-1))}{n^k} = 1(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{k-1}{n})$ approaches 1. Thus, there are $M_k \in \mathbb{N}$ for every $k \in \mathbb{N}$ s.t. $n \geqslant M_k \implies |\frac{n(n-1)\cdots(n-(k-1))}{n^k} - 1| < \frac{\epsilon}{2(N+1)\exp(R)}$. Taking $M := \max\{N, M_1, \ldots, M_N\} \in \mathbb{N}$, we have that for $n \geqslant M$ and our z fixed:

$$\left|\left(1+\frac{z}{n}\right)^n - \exp(z)\right| \leqslant \exp(R) \cdot \sum_{k=0}^{N} \frac{\epsilon}{2(N+1)\exp(R)} + \frac{\epsilon}{2} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

and so by the definition of limit, we have indeed proven that $\lim_{n\to\infty}(1+\frac{z}{n})^n = \exp(z)$. As we chose $z \in \mathbb{C}$ arbitrarily, the equality is true on all of $\mathbb{C}$, and we are done.

Exercise S&S §1.7

(Blaschke factors) Let $|a| \leqslant 1$ and $|b| \leqslant 1$ be.t. $\overline{a}b \neq 1 \iff \overline{\overline{a}b} = a\overline{b} \neq 1$. We want to show that $|\frac{a-b}{1-a\overline{b}}| \leqslant 1$ and examine when equality occurs. First note that the denominator can not be 0, since we are given $a\overline{b} \neq 1$. Recall that for any $z, w \in \mathbb{C}$, the conjugates of $z+w$, $z \cdot w$, and $\frac{1}{z}$ are respectively $\overline{z}+\overline{w}$, $\overline{z}\cdot\overline{w}$, and $\frac{1}{\overline{z}}$, and that $z\overline{z} = |z|^2$. Now, let us define $z = \frac{a-b}{1-a\overline{b}}$. The previous identities tells us that $\overline{z} = \frac{\overline{a}-\overline{b}}{1-\overline{a}b}$, and so

$$|z|^2 = z\overline{z} = \frac{a\overline{a} - b\overline{a} - \overline{b}a + b\overline{b}}{1 - a\overline{b} - \overline{a}b + a\overline{a}b\overline{b}} = \frac{|a|^2 - b\overline{a} - \overline{b}a + |b|^2}{1 - a\overline{b} - \overline{a}b + |a|^2|b|^2}.$$

Because a complex number times its conjugate is a square of a real number, both numerator and denominator are real and $\geqslant 0$, and since $|a|, |b|$, and 1 are all real, this means that $-b\overline{a} - \overline{b}a$ is a real number. Thus, to show that the numerator is $\leqslant$ the denominator, it suffices to show that $|a|^2 + |b|^2 \leqslant 1 + |a|^2|b|^2 \iff 1 - |a|^2 - |b|^2 + |a|^2|b|^2 \geqslant 0$. Well, we have that

$$|a|^2|b|^2 - |a|^2 - |b|^2 + 1 = (1-|a|^2)(1-|b|^2),$$

which is obviously $\geqslant 0$ since we assumed that $|a| \leqslant 1$ and $|b| \leqslant 1$. Furthermore, it is clear that it equals 0 if and only if $|a| = 1$ or $|b| = 1$ (inclusive or).

To **conclude**, we have shown that if $|a|, |b| < 1$ then we have a strict inequality $|\frac{a-b}{1-a\overline{b}}| < 1$ (i.e. numerator strictly < denominator), and that if at least one of $|a|, |b|$ are equal to 1, then $|\frac{a-b}{1-a\overline{b}}| = 1$ (i.e. numerator equals denominator).

Doing a variable name change $a \mapsto z, b \mapsto w$, let us now consider a fixed $w \in \mathbb{D}$ (the open unit disk, so $|w| < 1$) and the map $F : z \mapsto \frac{w-z}{1-\overline{w}z}$. First note that F is defined on all of $\overline{\mathbb{D}}$, since $|w| < 1, |z| \leqslant 1 \implies |\overline{w}z| = |w||z| < 1$, and so for $z \in \overline{\mathbb{D}}$ the denominator $(1 - \overline{w}z)$ will never be 0. We show that F satisfies the following properties:

(i) F maps the unit disk to itself, and is complex differentiable (=: "holomorphic"). Recall from the above **conclusion** that for $|z| < 1$ (note we fixed $|w| < 1$), $|F(z)| < 1$, so indeed $F(\mathbb{D}) \subseteq \mathbb{D}$. As for complex differentiability, F is simply a composition of complex addition/subtraction, multiplication, and division (where the denominator is non-zero on $\overline{\mathbb{D}}$, as discussed above), which are all complex differentiable functions, and hence F is also complex differentiable on $\mathbb{D}$.

(ii) F swaps 0 and w. Indeed we have $F(0) = \frac{w-0}{1-0} = w$, and $F(w) = \frac{w-w}{1-|w|^2} = 0$.

(iii) F maps $\partial\mathbb{D}$ to itself. Recall from the above **conclusion** that for $|z| = 1$ (note we fixed $|w| < 1$), $|F(z)| = 1$, so indeed $F(\partial\mathbb{D}) \subseteq \partial\mathbb{D}$.

(iv) $F : \mathbb{D} \to \mathbb{D}$ is bijective. We're given a hint to examine $F \circ F$. We do as the hint suggests: for $z \in \mathbb{D}$,

$$[F \circ F](z) = \frac{w - \frac{w-z}{1-\overline{w}z}}{1 - \overline{w}(\frac{w-z}{1-\overline{w}z})} = \frac{w(1-\overline{w}z) - (w-z)}{(1-\overline{w}z) - \overline{w}(w-z)} = \frac{z - |w|^2 z}{1 - |w|^2} = z,$$

so $[F \circ F] = \mathrm{id}$ on $\mathbb{D}$. Thus, F must be a bijection, since otherwise $F(\mathbb{D}) \subsetneq \mathbb{D} \implies \mathbb{D} = \mathrm{id}(\mathbb{D}) = [F \circ F](\mathbb{D}) = F(F(\mathbb{D})) \subseteq F(\mathbb{D}) \subsetneq \mathbb{D}$; contradiction.