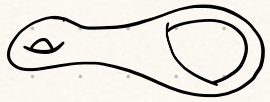
Complex Manifolds Geometry, Teichmuller space Visualization

6/20/25 oho ho, it's all coming together!

Σ_2 is notation for the set, which we should think of as a fixed embedded submanifold of $\mathbb{R}^3$.



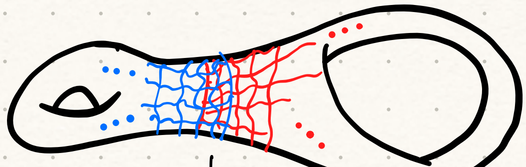
$\subseteq \mathbb{R}^3$

"complex str" = holomorphic atlas on Σ_2 should be pictured as patches w/ grid lines pasted all over Σ_2.

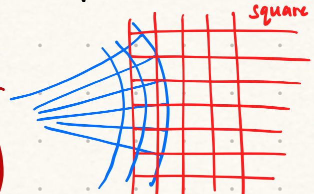
could be however crooked, but require that on overlaps, the (square) grid lines of one are smooth and form squares, as measured by the other grid.

(infinitesimal) + orientations match

"conformally compatible"



peel off patches and lay them flat (like peeling peel off orange and laying flat; but unlike that situation, here we don't care about isometry, so can deform peel.
However, must deform blue and red patch together
i.e. on overlap, any deformation you do to red patch will automatically deform blue patch along w/ it.)



"conformally compatible"

as drawn, blue grid definitely looks like local squares ("conformal") in eyes of red grid (= standard horizontal/vertical grid for this paper)

(vice versa by holomorphic inverse function theorem)

Can cut up our fixed Σ_2 into pairs of pants, which cut into 2 hexagonal pieces each

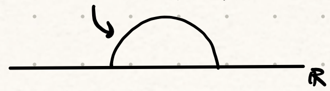
cut along "sides" (where seams of normal human pants are)

I will now describe how to get all complex structures on Σ_2 next.
First: have to give these hexagonal pieces a coordinate/grid patch.

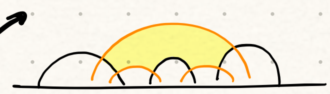
Pair of Pants

Consider (open) upper half plane $\mathbb{H} \subseteq \mathbb{C}$. Geodesics (with respect to the standard Poincare hyperbolic metric) are arcs of semicircles (endpts on $\mathbb{R}$), and straight vertical lines

$$(ds)^2 = \frac{(dx)^2 + (dy)^2}{y^2}$$



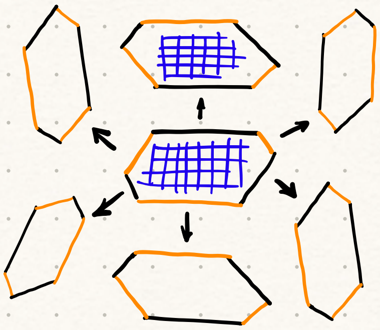
Form right-angled hexagons: sides are geodesics



Alex Wright "Tour through Mirzhikani's Work": "perhaps the most important miracle of low-dim. geometry" is that the group $PSL(2,\mathbb{R})$ of Möbius transformation stabilizing $\mathbb{H}$ is both – the group of orientation preserving isometries of $\mathbb{H}$
and – the group of biholomorphisms of $\mathbb{H}$

⇒ can reflect hexagon across its sides. (circle inversion)
↓
Möbius transformation (stabilizing $\mathbb{H}$)
so reflected copies of the hexagon are biholomorphic (but w/ flipped orientation)

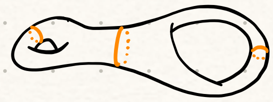
(drawn Euclidean-looking b/c I don't have the artistic skill for hyperbolic)



all 6 reflections $\in \mathbb{H}$ (w/ grid lines coming from horizontal/vertical grid lines in $\mathbb{H} \subseteq \mathbb{C}$)
are conformally equivalent
to each other (flipped conformal str of starting hexagon) orientation

Lemma: 2 right angled (geodesic) hexagons of same orientation in $\mathbb{H}$ are biholomorphic iff lengths of 3 alternating sides are equal
(i.e. length of 3 orange geodesic arcs) or 3 black
in std hyperbolic metric of $\mathbb{H}$

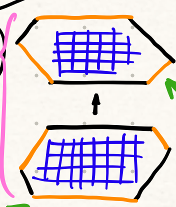
OK, recall we have



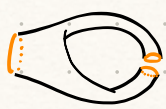
which consists of gluing 4 hexagonal pieces together along seams (glue along black seams to form pairs of pants; then glue along orange seams)

To put a complex str on Σ_2, first
we choose just 3 lengths (for the 3 orange seams)
(arbitrarily)

Now give each one of the 2 patches
(so that lengths follow and orientations can be glued)

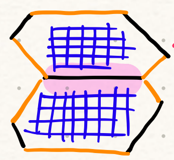


Example:



top piece give patch "orig"
and bottom piece give patch "↑"

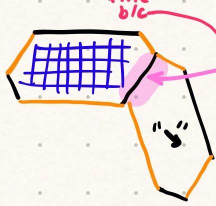
Can now give a conformally compatible patch on the upper black seam:



OK. but what about other seams? Middle black seam, lower black seam ?

It's actually easy: just pretend (fine b/c) that instead of choosing the patch "↑" for bottom hexagon,
we chose the patch "↘"

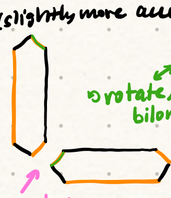
Then,



patch for middle black seam!
which is still conformally compatible w/ the actual patch "↑"
b/c "↘" and "↑" patches are conformally equivalent!

Do very similar things for orange seams

(slightly more accurate chosen lengths for my Σ_2)



in hyperbolic space, one way (maybe ∃ much better ways) is
rotate, which is biholomorphism
map center $c \in \mathbb{H} \to 0 \in \mathbb{D}$, rotate $e^{i\theta}$ apply to $\mathbb{D}$,
then transform back to $c \in \mathbb{H}$.

push together in $\mathbb{H}$, and steal that "collar" patch in $\mathbb{H}$ to be the patch covering the seam.

The above construction gives $\mathbb{R}^3_{>0}$ (3 lengths for 3 orange loops) many ("distinct") complex strs on Σ_2.

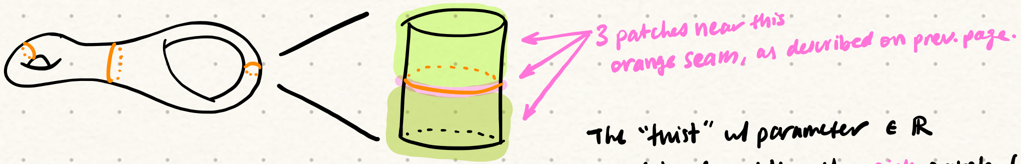
We now describe the move of "twisting" the 3 orange loops (above corresponds to twist = 0). Then the theorem is:

Thm: every complex structure on Σ_2 comes from deforming (isotopy from $id: \Sigma_2 \to \Sigma_2$) (i.e. patches moving as if dragged by) exactly 1 of the complex strs described above (i.e. $\exists!$ 3 lengths, $\exists!$ 3 twist values for 3 orange loops $\rightsquigarrow$ complex str as explicitly described above (and below))

(and of course analogous result for $g > 2$ also holds)

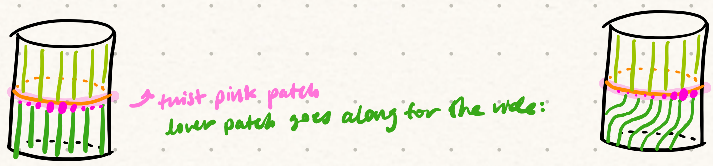
To conclude, let me explain "twisting".

Zoom in to an orange loop:

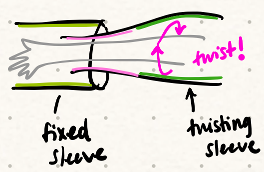


The "twist" w/ parameter $\in \mathbb{R}$ consists of twisting the pink patch (containing the orange loop) which drags **exactly one** of the nearby patches w/ it (say, drags the lower patch w/ it, but leaves upper patch as is!)

My attempt at illustrating:



Think putting one sleeve inside the other



and rotating your arms

for 3 nonseparating orange loops

nonseparating

cutting along it doesn't separate/disconnect Σ_2

nonseparating