Universal Property of Kernels & cokernels.

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Given $X \xrightarrow{f} Y$, kernel is $\overbrace{\text{universal obj}}^{\text{unique up to unique isomorphism}}$ satisfying

$$\begin{tikzcd} \mathrm{Ker}(f) \arrow[r, "\ker(f)"] & X \arrow[r, "f"] & Y \end{tikzcd},$$

namely $\forall$

test object

$$\begin{tikzcd} T \arrow[r] \arrow[d, dashed, "\exists!"'] \arrow[dr, phantom, "\equiv"] & X \arrow[r, "f"] & Y \\ K \arrow[ur] & & \end{tikzcd}$$

brown highlight means 0

$\equiv$ means diagram commutes

(UP)

Univ prop of mono & epi

$X \to Y$ is <u>epi</u> ($\twoheadrightarrow$) $:\Longleftrightarrow$

$$\left[\forall \; X \twoheadrightarrow Y \rightrightarrows T \text{ 2 yellow composition morphisms equal} \Rightarrow X \twoheadrightarrow Y \rightrightarrows T \text{ equal} \right]$$

this implication often annotated by "UP mono"

Corollary:

$$X \twoheadrightarrow Y \to Z \;\Rightarrow\; X \twoheadrightarrow Y \to Z$$

$\boxed{\text{Dual for mono. E.g.:}}$

Prop: $\mathrm{Ker}(f) \hookrightarrow X \xrightarrow{f} Y$

Pf:

$$\begin{tikzcd} T \arrow[r, shift left] \arrow[r, shift right] \arrow[rr, dashed, bend left, "\text{composition}"] & K \arrow[r] & X \arrow[r] & Y \end{tikzcd}$$

all green morphisms are equal

$$\begin{tikzcd} T \arrow[r, shift left] \arrow[r] \arrow[r, shift right, dashed, "\exists!"'] \arrow[rr, dashed, bend left] & K \arrow[r] & X \arrow[r] & Y \end{tikzcd}$$

! <u>uniqueness</u> forces all 3 arrows $T \rightrightarrows K$ equal!

$\boxed{\text{Dual for epi.}}$

Key Iso for Abelian Categories

$$X \xrightarrow{f} Y$$

Key Iso for Abelian Categories

kernel

cokernel

$$K_{XY} \hookrightarrow X \xrightarrow{f} Y \to Q_{XY}$$

pink = "coming from ker/coker"

Later, we will understand pink = "exact"

Key Iso for Abelian Categories

$$
\begin{tikzcd}
K_{XY} \arrow[dr, hook] & & & Q_{XY} \\
& X \arrow[r, "f"] \arrow[d] & Y \arrow[ur] & \\
& Q_{KX} & K_{YQ} \arrow[u] &
\end{tikzcd}
$$

Key Iso for Abelian Categories

$$\begin{tikzcd}
K_{XY} \arrow[dr, hook] & & & Q_{XY} \\
& X \arrow[r, "f"] \arrow[d] & Y \arrow[ur] & \\
& Q_{KX} \arrow[ur, dashed, "\cong", "\exists! \text{ (UP coker)}"'] & K_{YQ} \arrow[u, hook] &
\end{tikzcd}$$

Key Iso for Abelian Categories

$$\begin{tikzcd}
K_{XY} \arrow[dr, bend right] & & & Q_{XY} \\
& X \arrow[r, "f"] \arrow[d] & Y \arrow[ur] & \\
& Q_{KX} \arrow[ur, "\cong"] & K_{YQ} \arrow[u] &
\end{tikzcd}$$

brown arrow 1
means 1 = 0.
↓
1

Key Iso for Abelian Categories

$$
\begin{tikzcd}
K_{XY} \arrow[dr] & & & Q_{XY} \\
& X \arrow[r, "1\quad f"] \arrow[d, "2"'] \arrow[dr, phantom, "\cong"] & Y \arrow[ur] & \\
& Q_{KX} \arrow[ur] & K_{YQ} \arrow[u] &
\end{tikzcd}
$$

brown arrow ① means ①=0.

↓

① ⇒ ②

Key Iso for Abelian Categories

$$
\begin{tikzcd}
K_{XY} \arrow[dr] & & & Q_{XY} \\
& X \arrow[r, "f\ (1)"] \arrow[d, "(2)"'] & Y \arrow[ur] & \\
& Q_{KX} \arrow[ur, "(3)"'] & K_{YQ} \arrow[u] &
\end{tikzcd}
$$

brown arrow ① means ①=0.

① ⇒ ② ⇒ ③

UPepi

Key Iso for Abelian Categories

$$
\begin{tikzcd}
K_{XY} \arrow[dr] & & & Q_{XY} \\
& X \arrow[r, "f", "1"'] \arrow[d, "2"'] & Y \arrow[ur] & \\
& Q_{KX} \arrow[ur, red, "\cong"] \arrow[r, dashed, red, "4\ \exists!\ \text{UP ker}"'] & K_{YQ} \arrow[u] &
\end{tikzcd}
$$

3 $\cong$

brown arrow ① means ①=0.

$$① \underset{\cong}{\Rightarrow} ② \underset{\text{UPepi}}{\Rightarrow} ③ \underset{\text{UP ker}}{\Rightarrow} ④$$

Key Iso for Abelian Categories

$$
\begin{tikzcd}
K_{XY} \arrow[dr, bend right] & & & Q_{XY} \\
& X \arrow[r, "f"] \arrow[d] \arrow[r, phantom, shift right=6, "\cong"] & Y \arrow[ur] & \\
& Q_{KX} \arrow[ur] \arrow[r, "4"'] \arrow[r, phantom, shift left=6, "\cong"] & K_{YQ} \arrow[u] &
\end{tikzcd}
$$

Axiom of Ab Cat, red arrow is isomorphism, can consider it one object (called image of $X \xrightarrow{f} Y$) (w/ 2 canonical maps)

Key Iso for Abelian Categories

if:

$$
\begin{tikzcd}
\overset{\text{kernel}}{K_{XY}} \arrow[r, hook] & X \arrow[r, "f"] & Y \arrow[r, two heads] & \overset{\text{cokernel}}{Q_{XY}}
\end{tikzcd}
$$

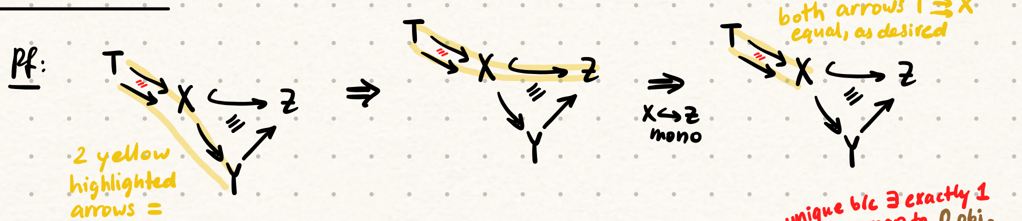
then: $\Downarrow$

$$
\begin{tikzcd}
K_{XY} \arrow[r, hook] & X \arrow[r, "f"] \arrow[dr, two heads] & Y \arrow[r, two heads] & Q_{XY} \\
& & I_{XY} \arrow[u, hook] &
\end{tikzcd}
$$

More on mono, epi, ker, coker

Prop:
$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, hook] & Z \\ & Y \arrow[u] \arrow[ul, phantom, "\equiv"] \end{tikzcd}$$
must be mono

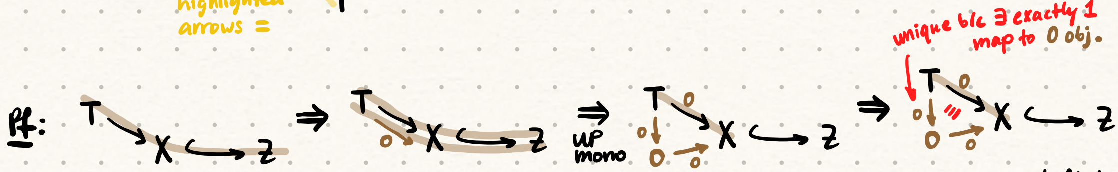
Pf:



3/11/26

Prop: Kernel of mono is 0

Pf:

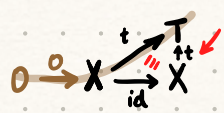


so $0 \xrightarrow{0} X$ satisfies UP for $\ker(X \hookrightarrow Z)$

Converse: $\ker = 0 \Rightarrow$ mono. Pf: $T \overset{f}{\underset{g}{\rightrightarrows}} X \to Y$. Take difference of the 2 maps $f-g$: $T \xrightarrow{f-g} X \to Y$. Then $\xrightarrow{f-g}$ factors thru $0 \Rightarrow f-g=0$

Prop: cokernel of $0 \xrightarrow{0} X$ is $X \xrightarrow{id} X$

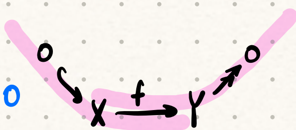
Pf:



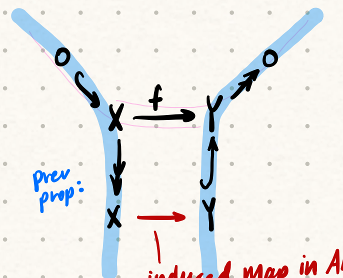
unique map s.t. $\begin{tikzcd} \cdot \arrow[r, "t"] \arrow[dr, "id"'] & \cdot \\ & \cdot \arrow[u] \end{tikzcd}$ commutes.

so by UP cok, $\xrightarrow{id} X$ is cok.

Prop: $\ker = \mathrm{Cok} = 0 \Rightarrow$ iso.



(i.e. $\ker(f) = 0 = \mathrm{Cok}(f)$)



Exactness

Thm: Given $\underbrace{X \hookrightarrow}_{=} Y \underbrace{\twoheadrightarrow Z}_{=}.$

equal up to isom $\quad K_{YZ} \hookrightarrow \iff \twoheadrightarrow Q_{XY}$

Pf: $(\Leftarrow)$

$$\begin{tikzcd}
K_{XY}=0 \arrow[d, hook, "0"'] & & & Q_{XY} \\
X \arrow[r, hook] & Y \arrow[urr] & &
\end{tikzcd}$$

prop on prev slide

$\Rightarrow$

$$\begin{tikzcd}
K_{XY}=0 \arrow[d, hook, "0"] & & Q_{XY} \\
X \arrow[r, hook] \arrow[d, "\mathrm{id}"'] & Y \arrow[ur] & \\
X \arrow[r, leftrightarrow, "\text{isom by AbCat axiom}"'] & K_{YQ} \arrow[u, hook] &
\end{tikzcd}$$

prop on prev slide

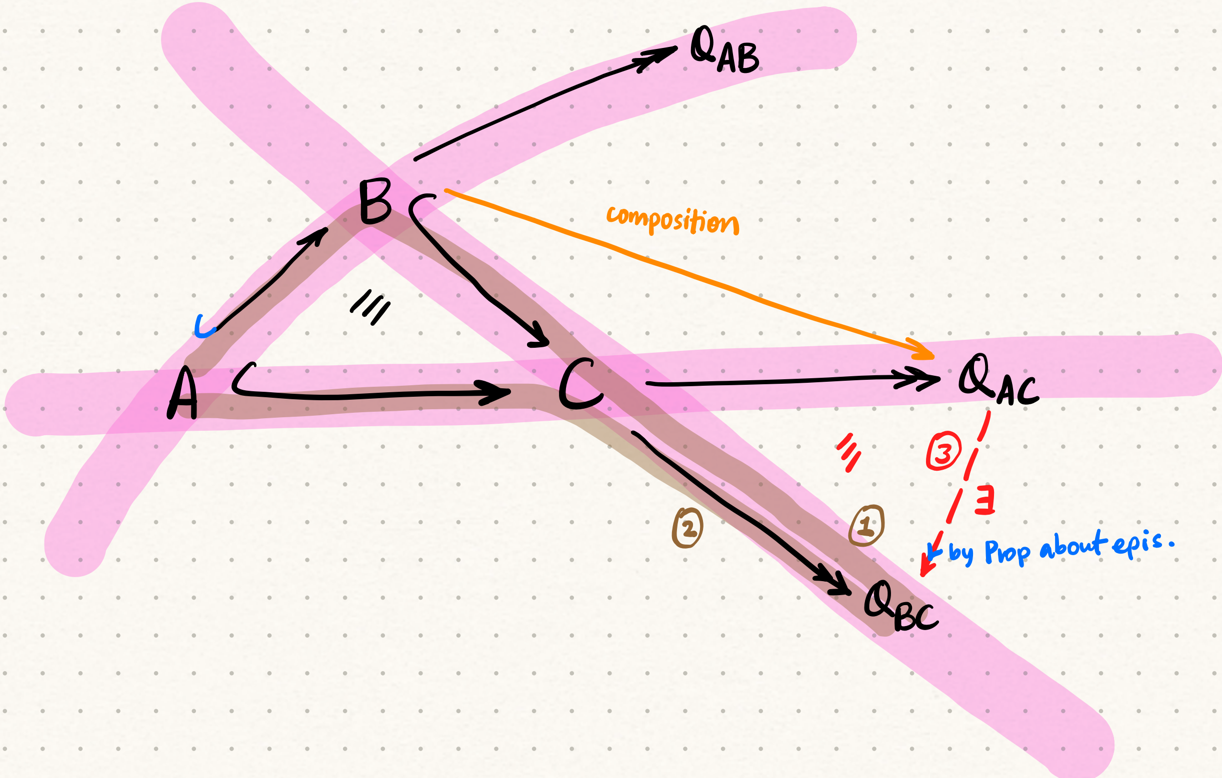
Noether Isomorphism. (cf. Theo Bühler Lemma 3.5)

$$\begin{tikzcd} & B \arrow[dr, hook] & \\ A \arrow[ur, hook, "\text{Prop. on a prev slide}"] \arrow[rr, hook] & & C \end{tikzcd}$$

Noether Isomorphism. (cf. Theo Bühler Lemma 3.5)

$$
\begin{tikzcd}
 & & & Q_{AB} \\
 & B \arrow[urr] \arrow[dr, hook] \arrow[drr, "\text{composition (of 2 existing black arrows)}"] & & \\
A \arrow[ur, hook] \arrow[rr, hook] & & C \arrow[r] \arrow[dr] & Q_{AC} \\
 & & & Q_{BC}
\end{tikzcd}
$$

Noether Isomorphism. (cf. Theo Bühler Lemma 3.5)



① = ② $\Rightarrow$ ③
UP
cok

Noether Isomorphism. (cf. Theo Bühler Lemma 3.5)

$$
\begin{tikzcd}
 & & & Q_{AB} & \\
 & B \arrow[urr] \arrow[dr, hook] \arrow[drrr, "\text{composition}"] & & & \\
A \arrow[ur, hook] \arrow[rr, hook] & \arrow[r, phantom, "\cong"] & C \arrow[rr, two heads] \arrow[drr, two heads] & \arrow[dr, phantom, "\cong"] & Q_{AC} \arrow[d, dashed] \\
 & & & & Q_{BC}
\end{tikzcd}
$$

Claim: Q_{BC} is actually $\mathrm{cok}(B \to Q_{AC})$

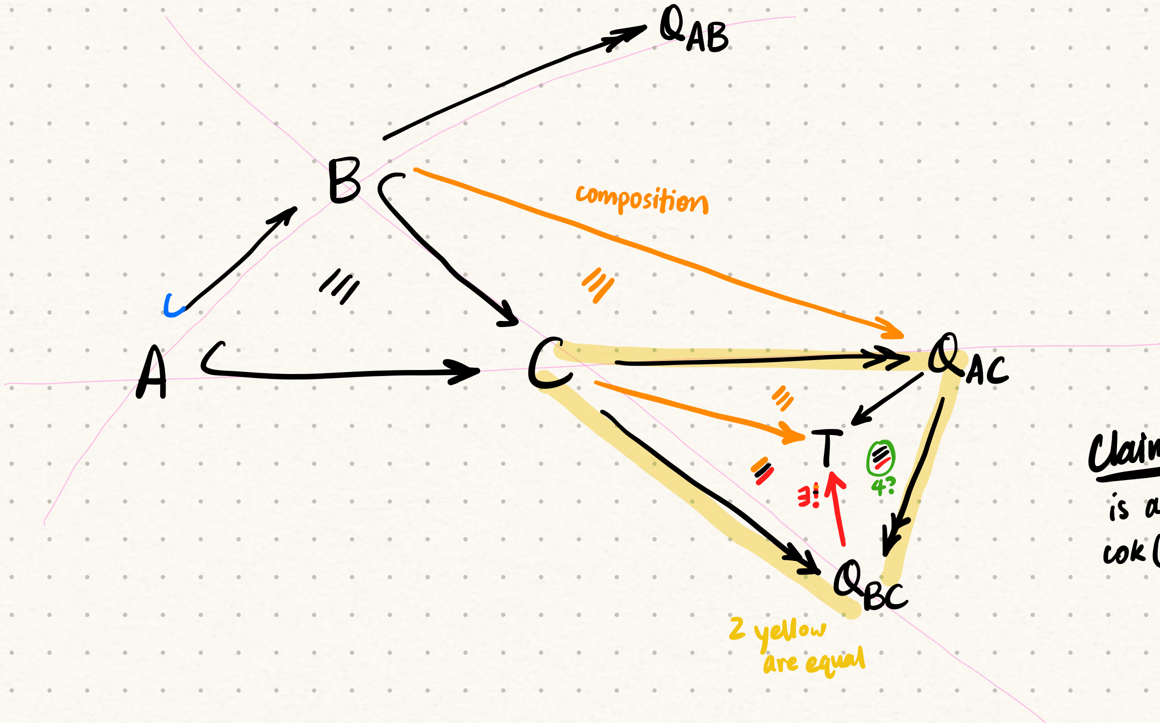
Noether Isomorphism. (cf. Theo Bühler Lemma 3.5)

$$
\begin{tikzcd}
 & & & Q_{AB} & \\
 & B \arrow[urr] \arrow[dr, hook] \arrow[drr, "\text{composition}"] & & & \\
A \arrow[ur, hook] \arrow[rr, hook] & & C \arrow[r] \arrow[dr] \arrow[drr, bend right, "②"'] & Q_{AC} \arrow[dr, "①"] \arrow[d, "\equiv\ ④?"'] & \\
 & & & Q_{BC} \arrow[r, "\exists!\ ③"'] & T
\end{tikzcd}
$$

Claim: Q_{BC} is actualy $\operatorname{cok}(B \to Q_{AC})$

$① = ② \overset{\text{up}}{\underset{\text{cok}}{\Rightarrow}} ③$ Remains to show ④

Noether Isomorphism. (cf. Theo Bühler Lemma 3.5)



Claim: Q_{BC} is actualy $\mathrm{cok}(B \to Q_{AC})$

① = ② $\underset{\text{up cok}}{\Rightarrow}$ ③ Remains to show ④

Noether Isomorphism. (cf. Theo Bühler Lemma 3.5)

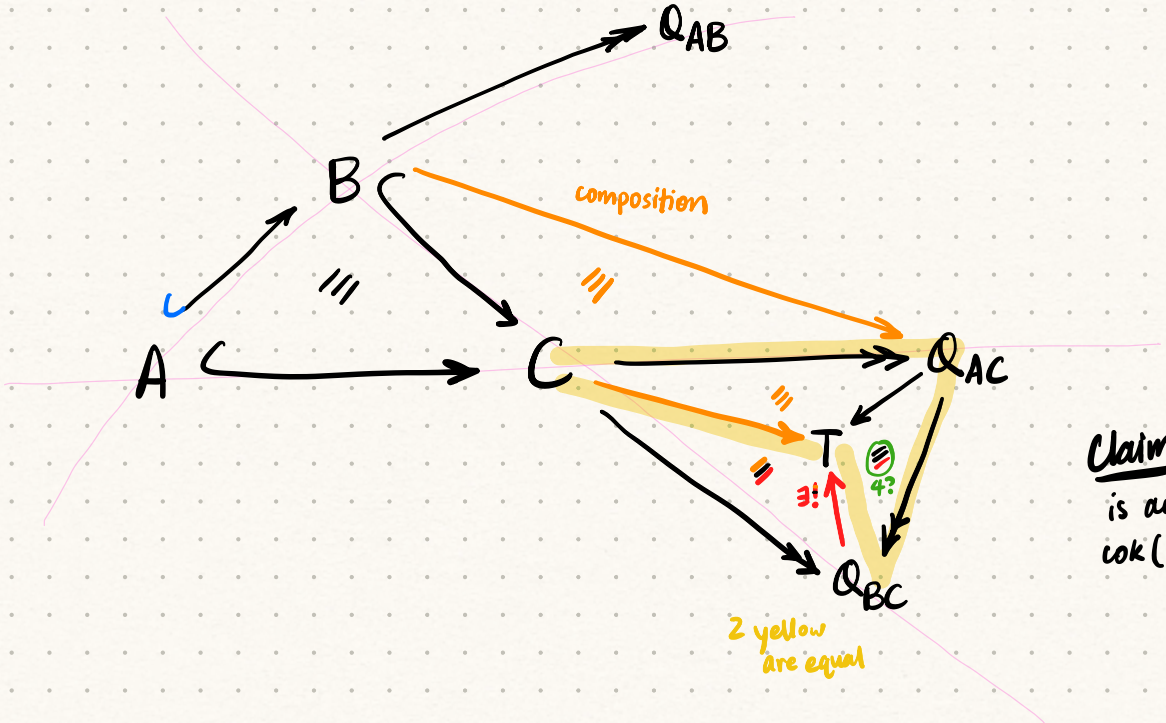
$$\begin{tikzcd}
 & & & & Q_{AB} \\
 & B \arrow[urrr] \arrow[dr, hook] \arrow[drrr, "\text{composition}"] & & & \\
A \arrow[ur, hook] \arrow[rr, hook] & & C \arrow[rr] \arrow[dr] \arrow[ddrr] & & Q_{AC} \arrow[dl] \arrow[dd] \\
 & & & T & \\
 & & & & Q_{BC} \arrow[ul, "\exists!"]
\end{tikzcd}$$

2 yellow are equal

Claim: Q_{BC} is actually $\mathrm{cok}(B \to Q_{AC})$

① = ② $\underset{\text{up cok}}{\Rightarrow}$ ③ Remains to show ④

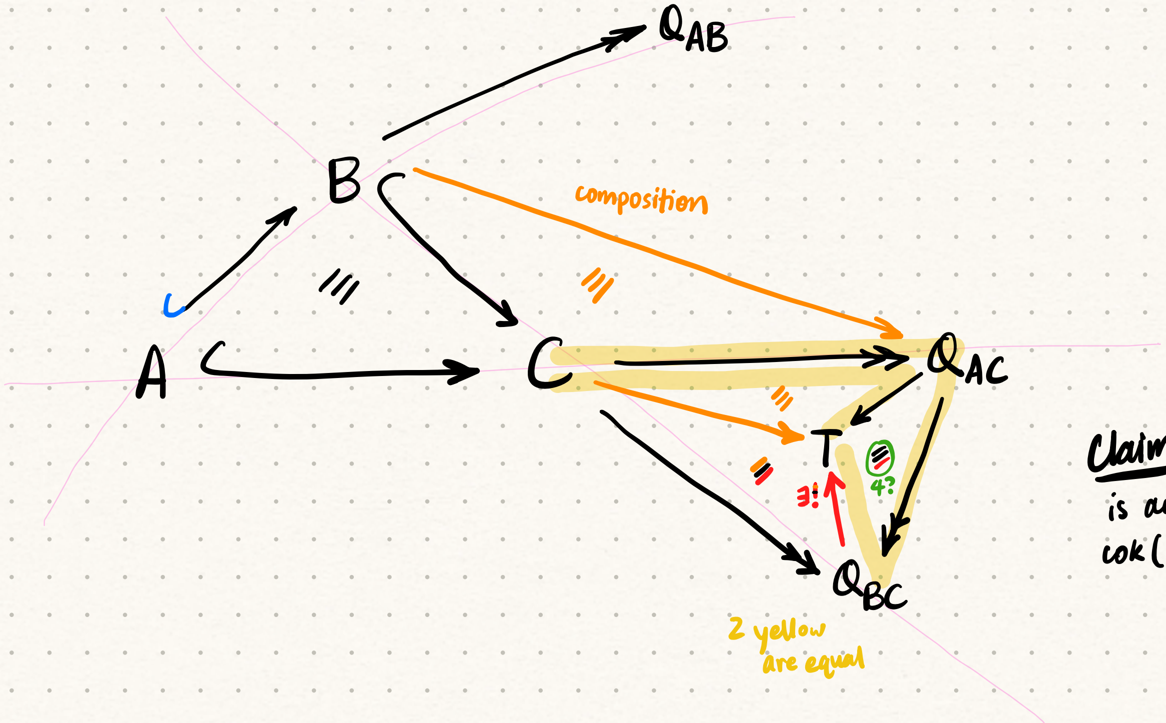
Noether Isomorphism. (cf. Theo Bühler Lemma 3.5)



Claim: Q_{BC} is actually $\mathrm{cok}(B \to Q_{AC})$

1 = 2 $\overset{\text{up cok}}{\Rightarrow}$ 3 Remains to show 4

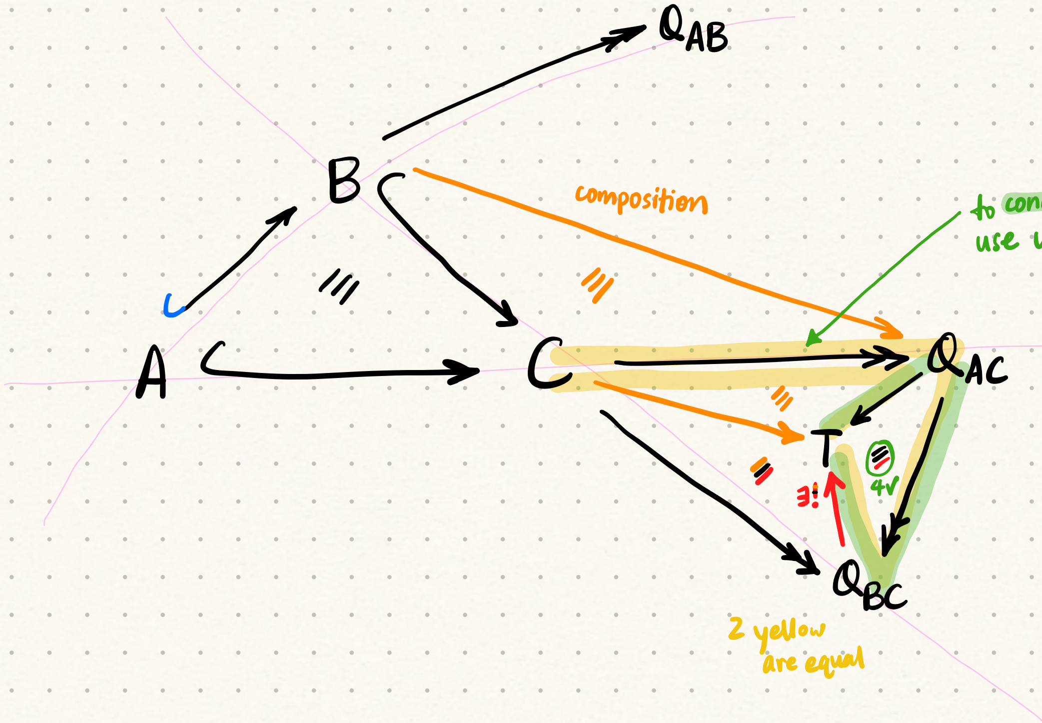
Noether Isomorphism. (cf. Theo Bühler Lemma 3.5)



Claim: Q_{BC} is actually $\mathrm{cok}(B \to Q_{AC})$

① = ② $\underset{\text{up cok}}{\Rightarrow}$ ③ Remains to show ④

Noether Isomorphism. (cf. Theo Bühler Lemma 3.5)



to conclude pf of ④
use UP of epi !!!

✓ **Claim**: Q_{BC}
is actually
cok($B \to Q_{AC}$)

① = ② $\Rightarrow$ ③
up
cok

✓
Remains
to show ④

Noether Isomorphism. (cf. Theo Bühler Lemma 3.5)

$$
\begin{tikzcd}
 & B \arrow[rr] \arrow[dr, hook] \arrow[drrr, "\text{composition}"] & & Q_{AB} & \\
A \arrow[ur, hook] \arrow[rr, hook] & & C \arrow[rr] \arrow[dr] & & Q_{AC} \arrow[dl] \\
 & & & Q_{BC} &
\end{tikzcd}
$$

Claim: Q_{BC} is actually $\operatorname{cok}(B \to Q_{AC})$

Noether Isomorphism. (cf. Theo Bühler Lemma 3.5)

$$\begin{tikzcd}
 & B \arrow[rr] \arrow[dr, hook] \arrow[drrr, "\text{composition}"] & & Q_{AB} & \\
A \arrow[ur, hook] \arrow[rr, hook] & & C \arrow[rr] \arrow[drr] & & Q_{AC} \arrow[d] \\
 & & & & Q_{BC}
\end{tikzcd}$$

Similarly, can show $A \hookrightarrow B$ is $\ker(\rightarrow)$

Claim: Q_{BC} is actually $\operatorname{cok}(B \rightarrow Q_{AC})$

Then apply AbCat Iso! Axiom

Noether Isomorphism. (cf. Theo Bühler Lemma 3.5)

$$
\begin{tikzcd}
 & & & Q_{AB} \arrow[ddr] & \\
 & B \arrow[urr] \arrow[dr, hook] \arrow[drrr, "\text{composition}"] \arrow[d, phantom, "\cong"] & & & \\
A \arrow[ur, hook] \arrow[rr, hook] & & C \arrow[rr] \arrow[drr] \arrow[dr, phantom, "\cong"] & & Q_{AC} \arrow[d] \\
 & & & & Q_{BC}
\end{tikzcd}
$$

Then apply Ab Cat Iso! Axiom

QED.

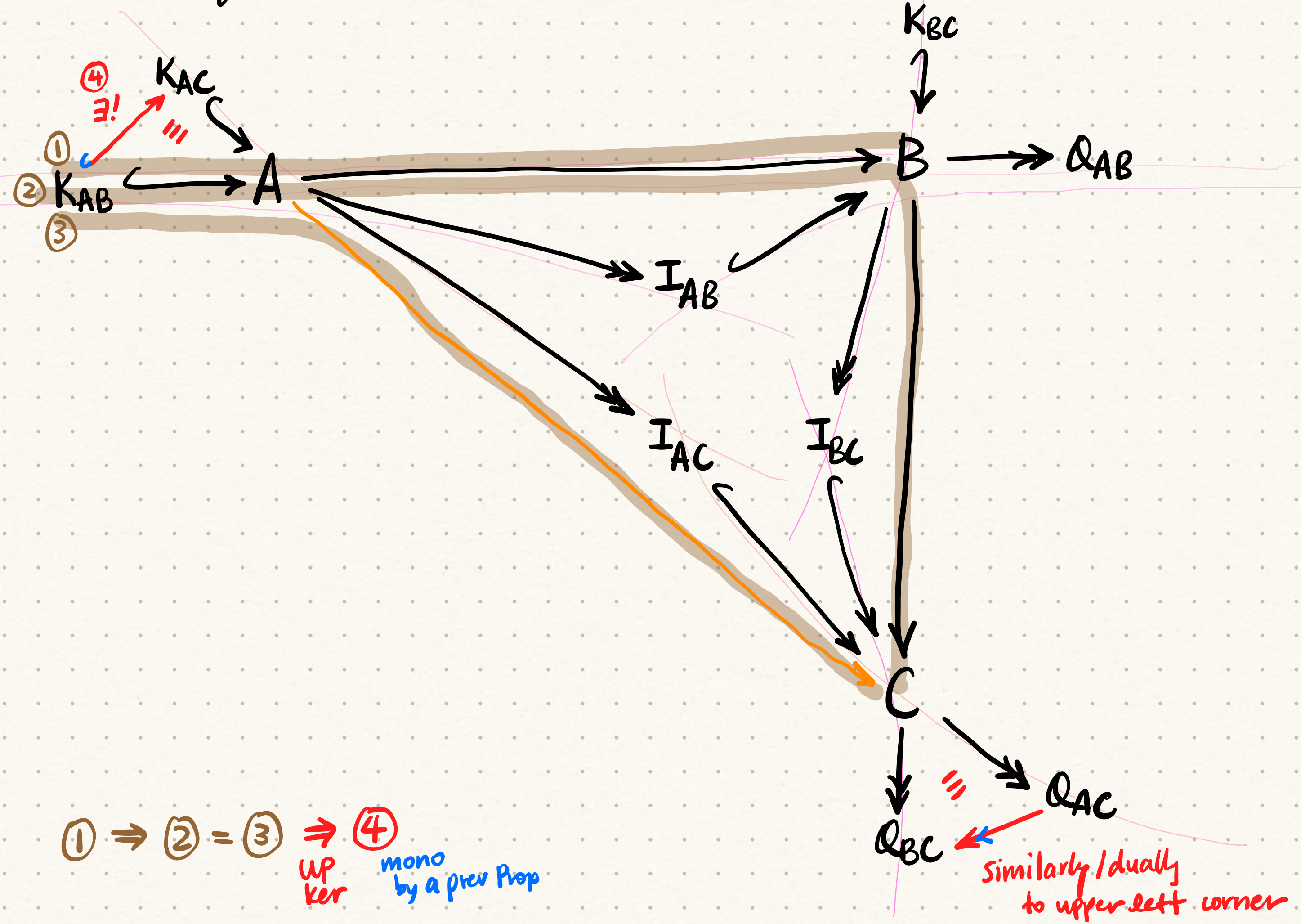
Ker Coker Sequence (cf. Theo Bühler Prop 8.11)

$$
\begin{tikzcd}
A \arrow[r] \arrow[dr] & B \arrow[d] \\
 & C
\end{tikzcd}
$$

Ker Coker Sequence (cf. Theo Bühler Prop 8.11)

$$
\begin{tikzcd}
K_{AC} \arrow[dr, hook] & & & & K_{BC} \arrow[d, hook] & \\
K_{AB} \arrow[r, hook] & A \arrow[rrr] \arrow[dr, two heads] \arrow[ddr, two heads] \arrow[dddrrr] & & & B \arrow[r, two heads] \arrow[ddl, two heads] \arrow[ddd] & Q_{AB} \\
 & & I_{AB} \arrow[urr, hook] & & & \\
 & & I_{AC} \arrow[drr, hook] & I_{BC} \arrow[dr, hook] & & \\
 & & & & C \arrow[r, two heads] \arrow[d, two heads] & Q_{AC} \\
 & & & & Q_{BC} &
\end{tikzcd}
$$

Ker Coker Sequence (cf. Theo Bühler Prop 8.11)

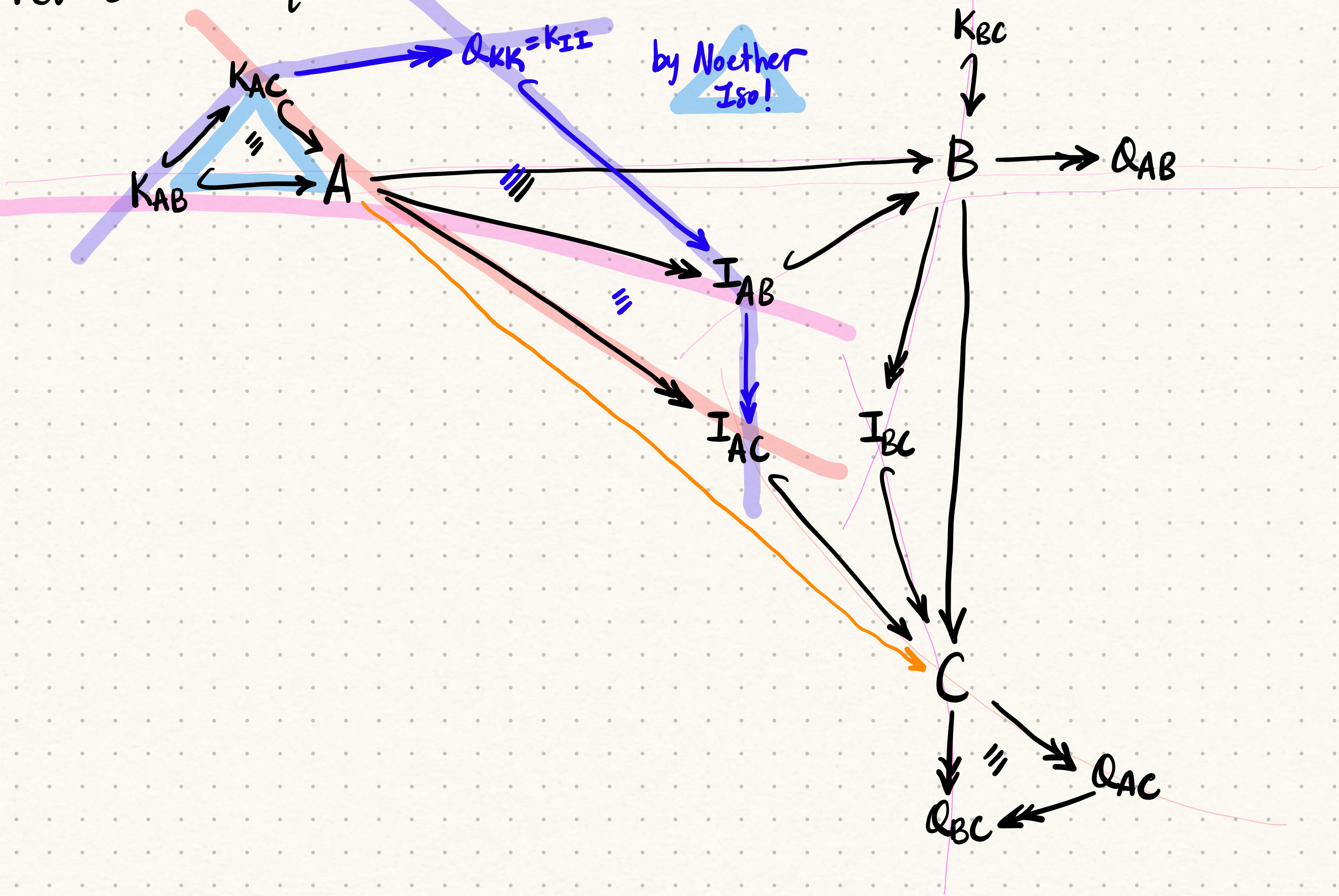


① ⇒ ② = ③ ⇒ ④

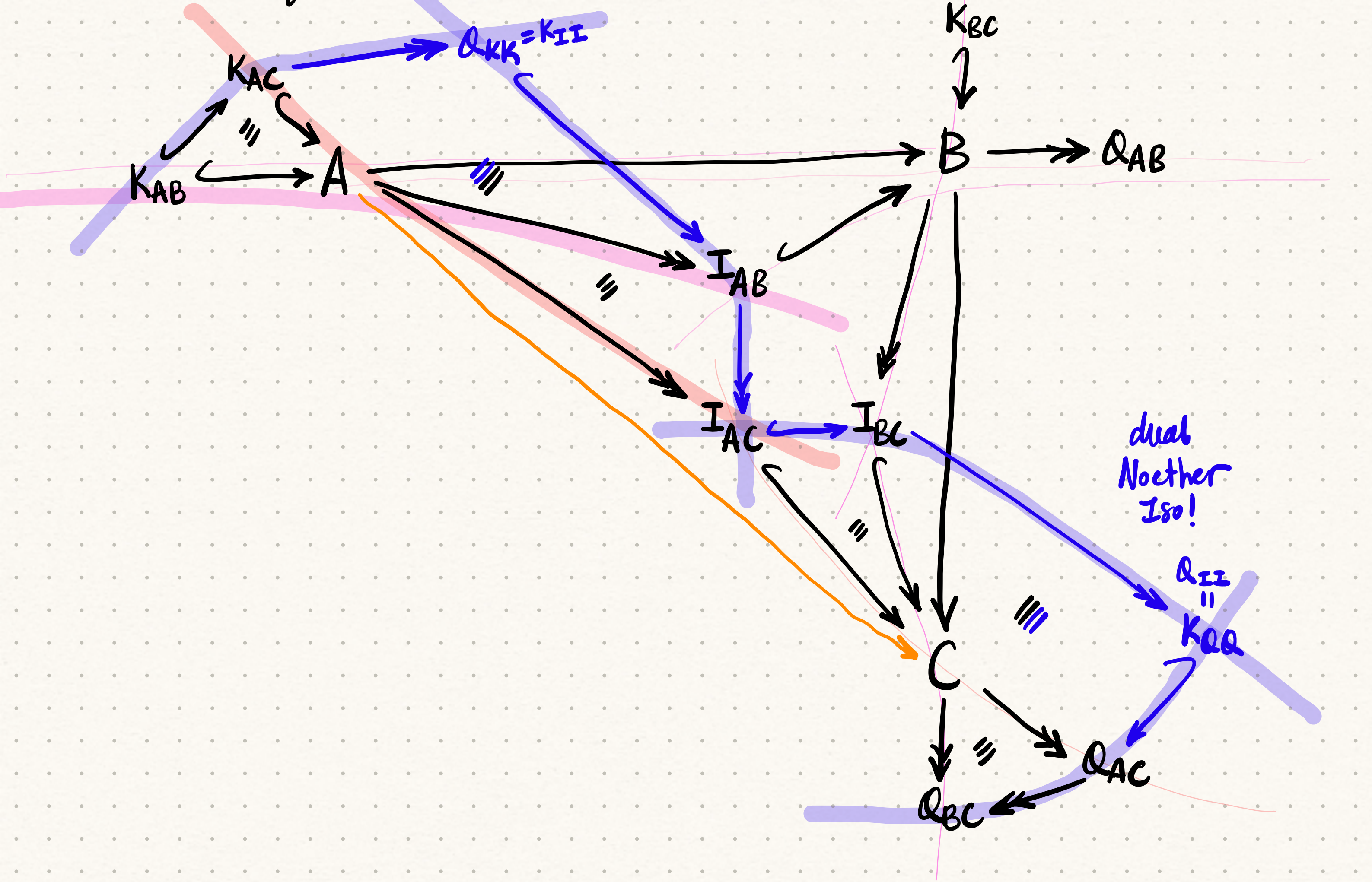
up ker

mono by a prev Prop

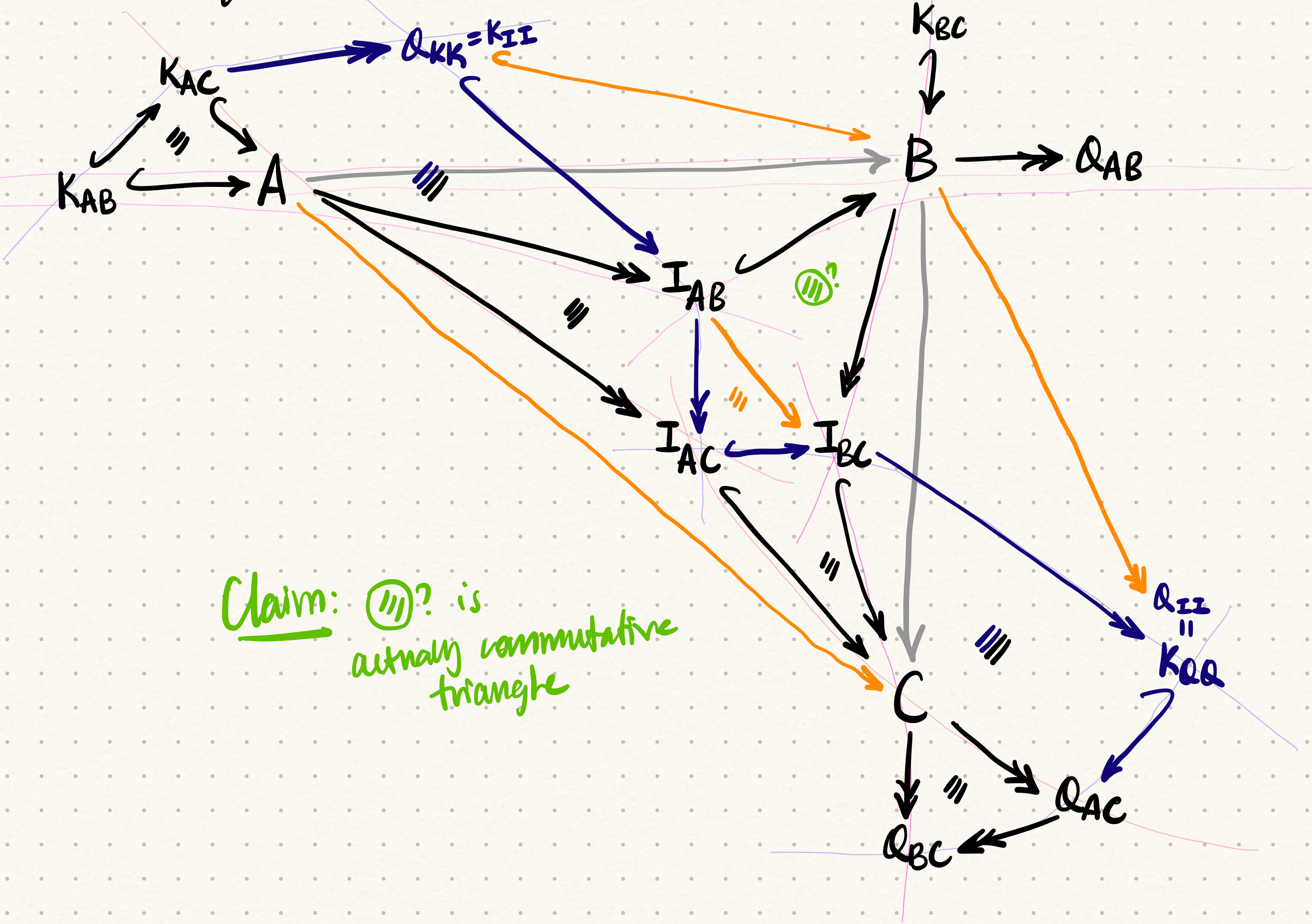
Ker Coker Sequence (cf. Theo Bühler Prop 8.11)



Ker Coker Sequence (cf. Theo Bühler Prop 8.11)

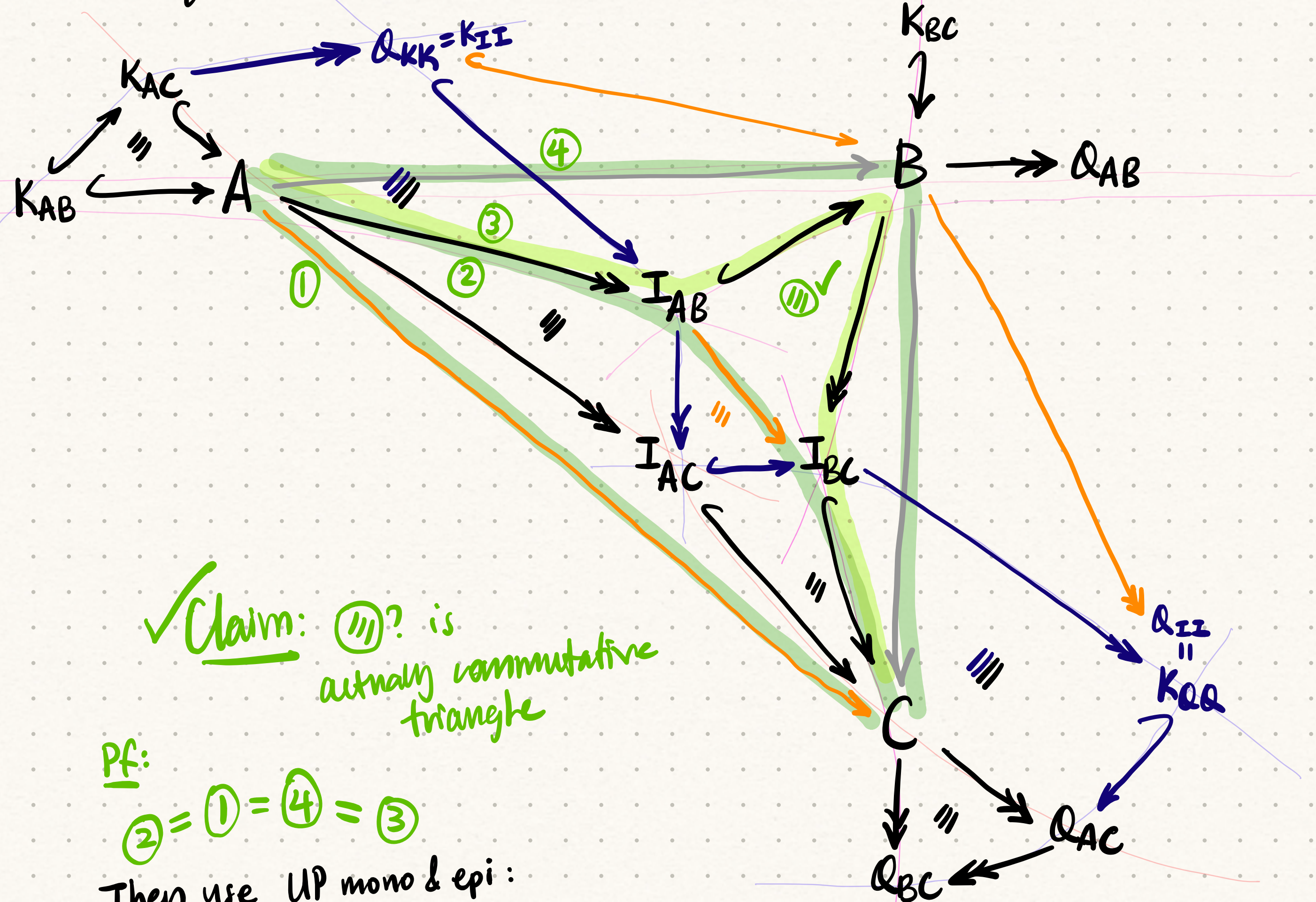


Ker Coker Sequence (cf. Theo Bühler Prop 8.11)



Claim: ///? is actually commutative triangle

Ker Coker Sequence (cf. Theo Bühler Prop 8.11)



✓ Claim: (///)? is actually commutative triangle

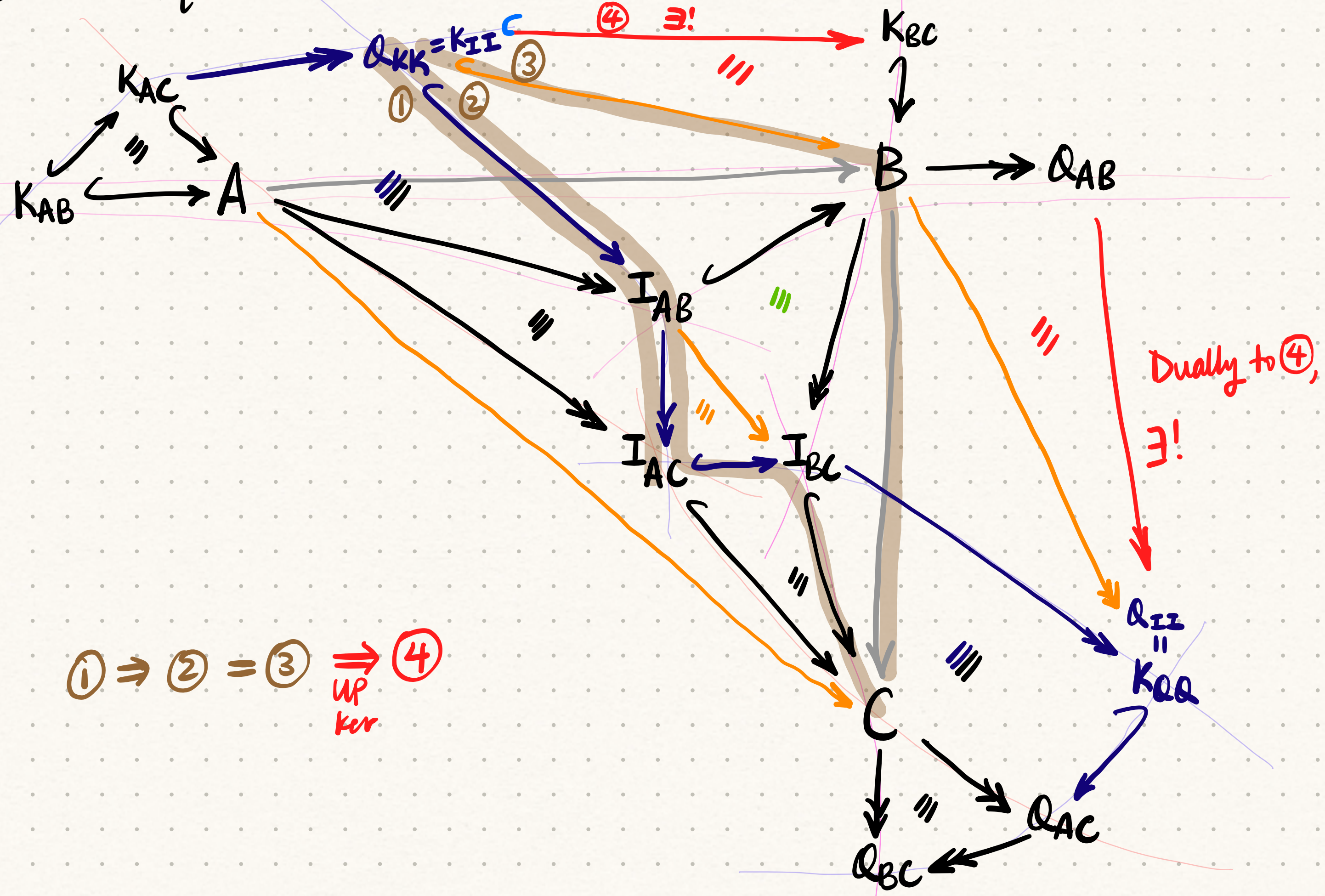
Pf:

② = ① = ④ = ③

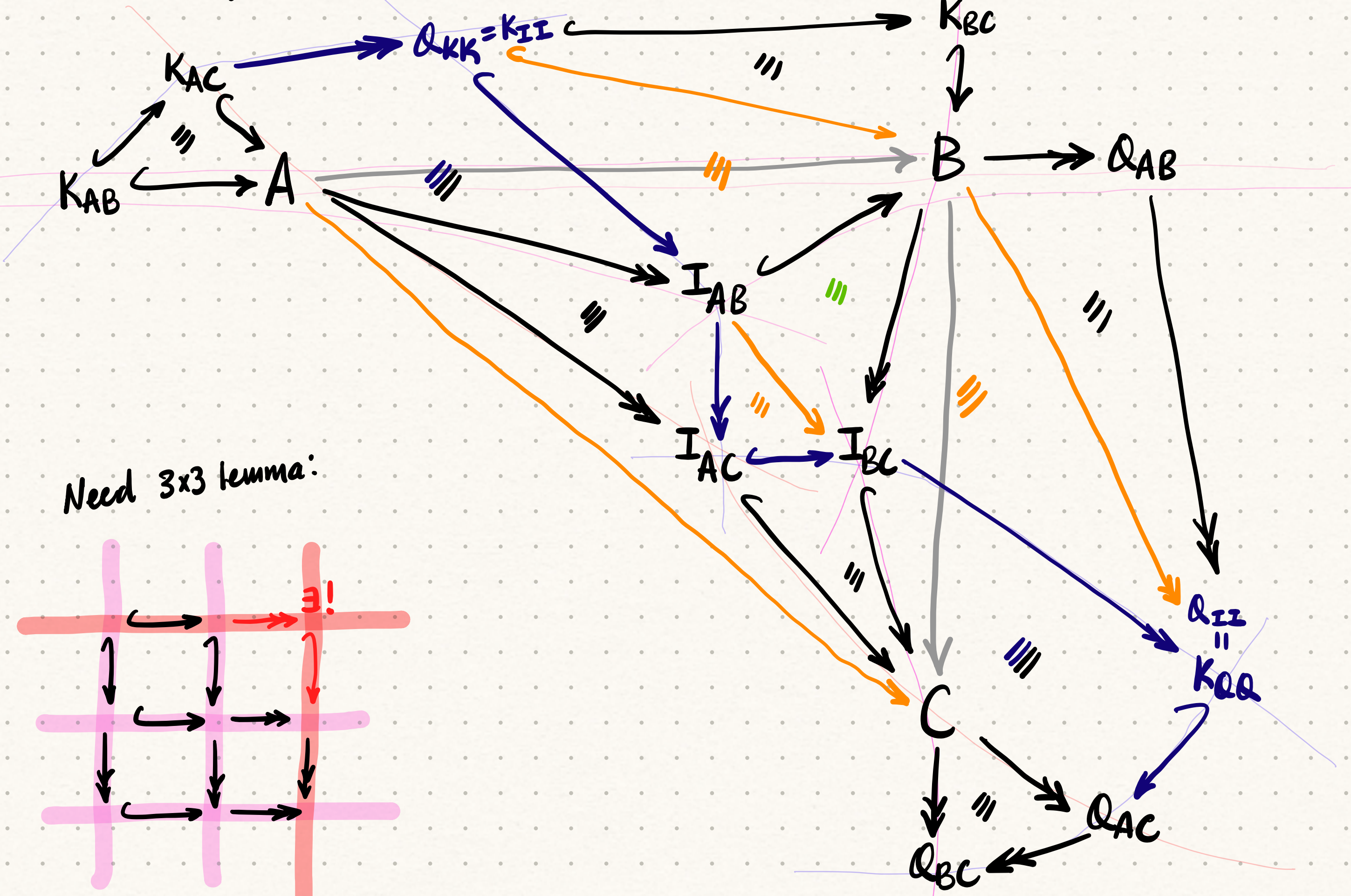
Then use UP mono & epi:

$\twoheadrightarrow \; \rightrightarrows \; \hookrightarrow \; \Rightarrow \; \rightrightarrows$

Ker Coker Sequence (cf. Theo Bühler Prop 8.11)



Ker Coker Sequence (cf. Theo Bühler Prop 8.11)

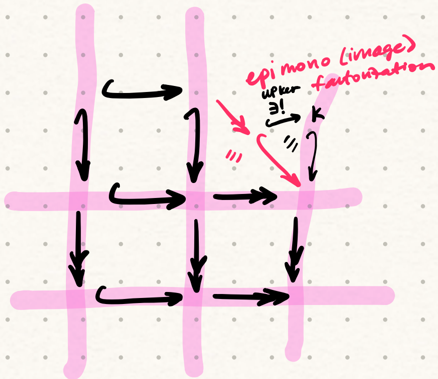


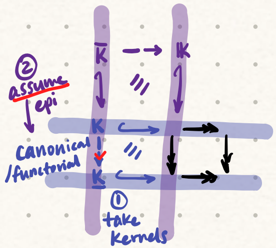
Need 3x3 lemma:

$$\begin{tikzcd}
\bullet \arrow[r, hook] \arrow[d] & \bullet \arrow[r, dashed] \arrow[d] & \exists! \arrow[d, dashed] \\
\bullet \arrow[r, hook] \arrow[d] & \bullet \arrow[r, two heads] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r, hook] & \bullet \arrow[r, two heads] & \bullet
\end{tikzcd}$$

But this follows from **Ab Cat Axiom**:

$$\begin{tikzcd} \arrow[r, hook] & I_{AB} \arrow[r, hook] & B \arrow[r] & Q_{AB} \end{tikzcd}$$

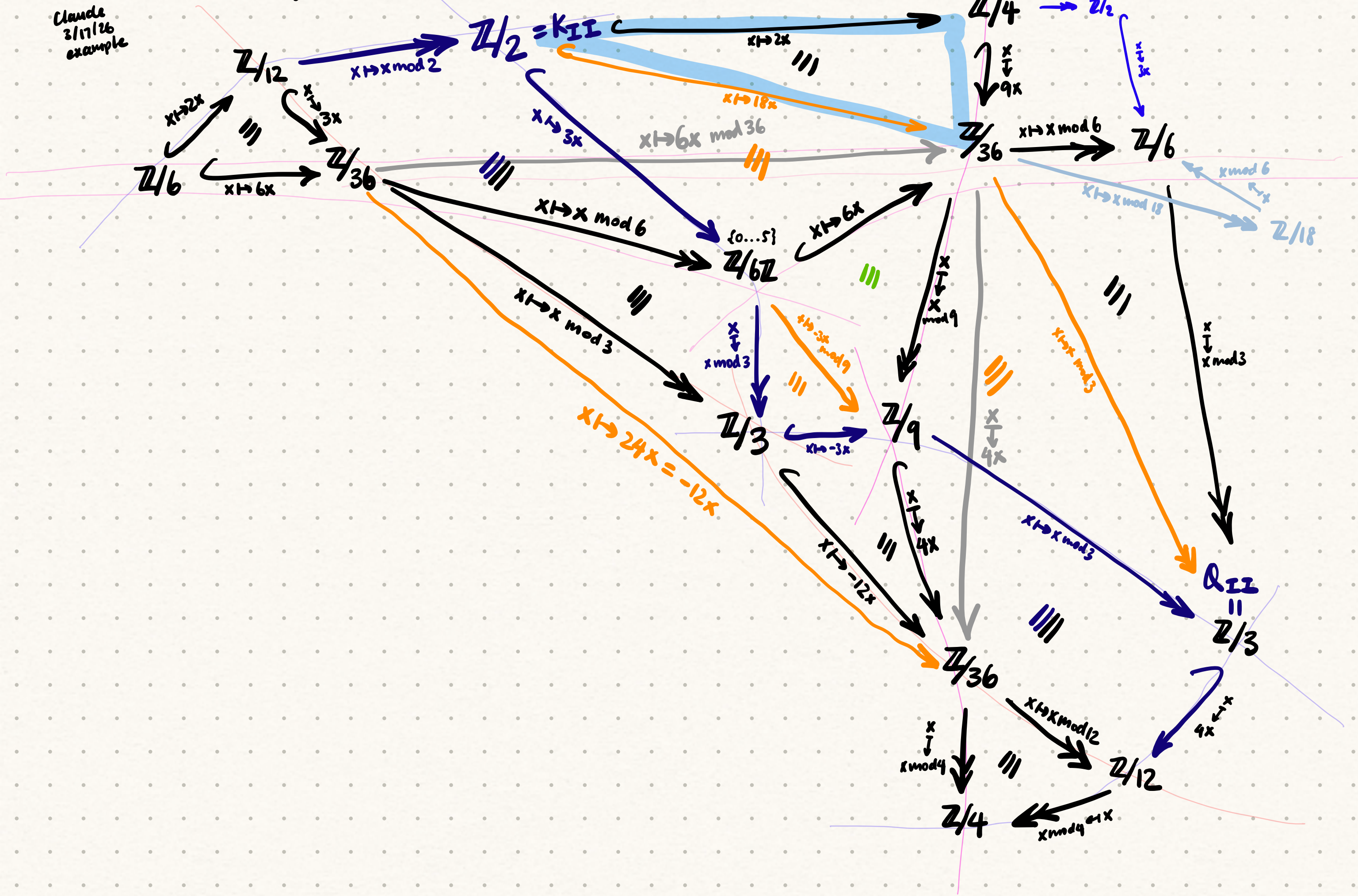




give up, continue
copying Balmer below
2BC / hexagon thm

Ker Coker Sequence (cf. Theo Bühler Prop 8.11)

Claude 3/17/26 example



Junk.

$$\begin{tikzcd}
 & B \arrow[dr, hook] & & \\
A \arrow[ur, hook] \arrow[rr, hook] & & C \arrow[r, hook] \arrow[dr] & Y \arrow[d] \\
 & K_{CY} \arrow[ur, hook] & & Z
\end{tikzcd}$$

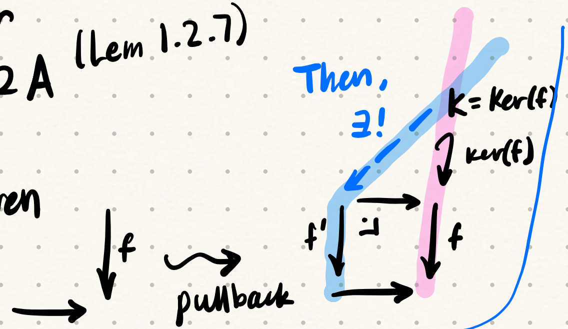
~~To apply NoetherIso~~
~~one last time,~~
~~need~~ $\cancel{Q_{KK} \hookrightarrow B = \ker(B \twoheadrightarrow Q_{AB})}$

below 3/18/26

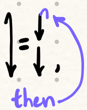
Liu's notes on Balmer 212A (Lem 1.2.7)

"Cartesian ⇒"

Prop: Given



Lemma: Given



Pf: $T \rightrightarrows X \hookrightarrow \to Y$

$\Downarrow$

$T \rightrightarrows X \hookrightarrow \to Y$

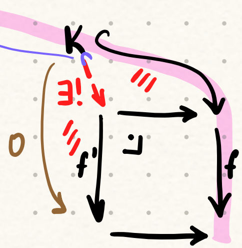
$\Downarrow$

$T \rightrightarrows X \hookrightarrow \to Y$

$\Downarrow$ up mono

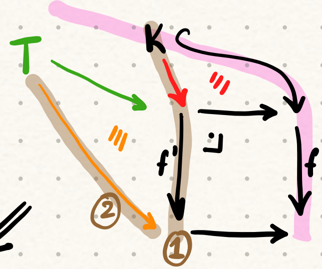
$T \rightrightarrows X \hookrightarrow \to Y$

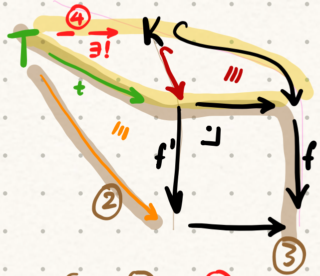
Pf:



by UP ⌟

Consider test object T s.t. $\to$:





② ⇒ ③ ⇒ ④ UP ker satisfying commutativity properties

Cor: Given

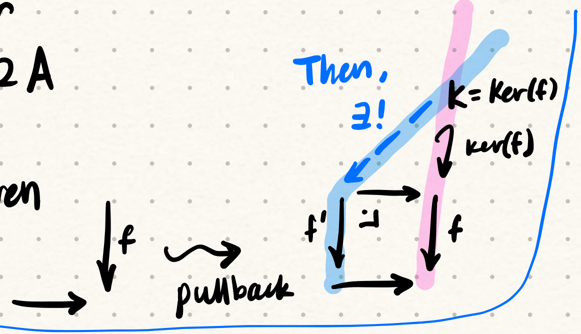
(Liu pg 13 footnote 2, says this ("pullback of mono is mono") holds ∀ category, not just Abelian)

Cor: dual to prev cor (above pf)

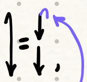
pullback of epi is epi

Liu's notes on Balmer 212A

Prop: Given

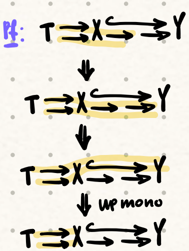


Lemma: Given

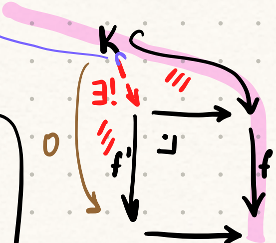


then

Pf:

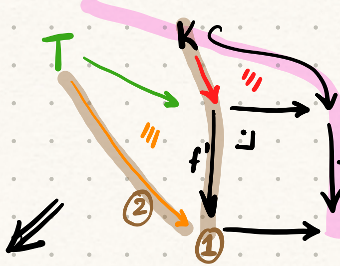


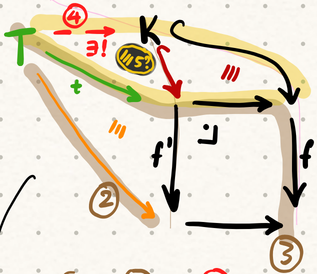
Pf:



by UP ⌟

Consider test object T s.t. :



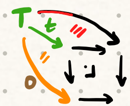


② ⇒ ③ ⇒ ④

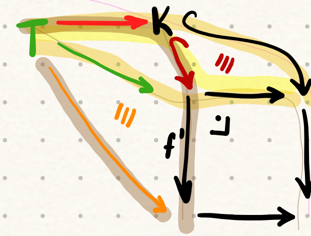
UP ker

satisfying commutativity properties

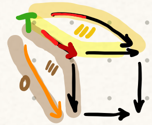
in ptclr, this diagram says



But also



says

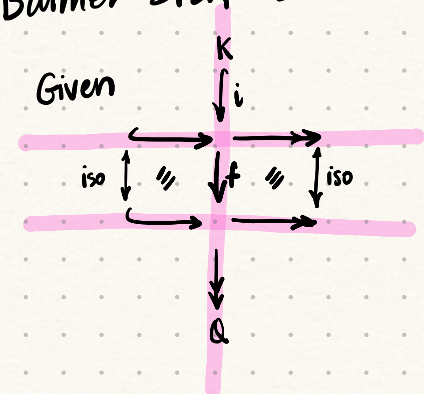


so by uniqueness in ⌟UP,

Finally, ↘ unique s.t.  b/c ↪ mono!

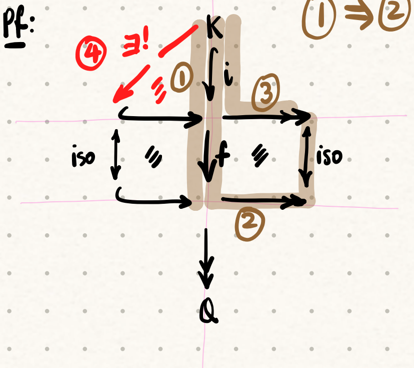
Balmer 212A Liu notes Thm 1.4.5 Five Lemma special case

Given



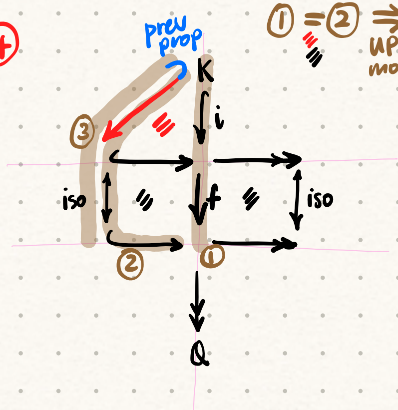
then f is iso.

Pf:



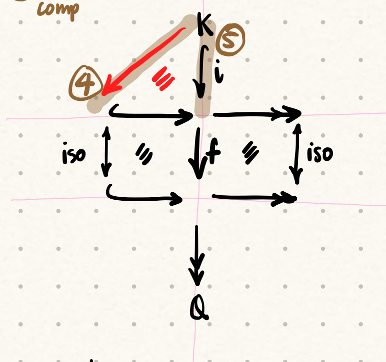
① ⇒ ② = ③ ⇒ ④
UP ker

⇒



① = ② ⇒ ③ ⇒ ④ ⇒ ⑤ ⇒ K=0!
UP mono
b/c iso
comp

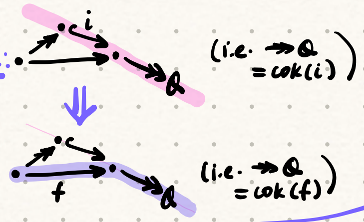
⇒



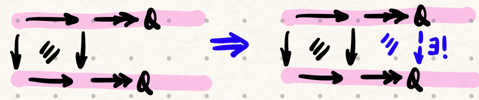
Dual for $\downarrow Q$.

Next slide: Five Lemma General Case.

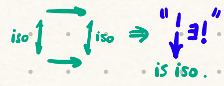
Lemma:

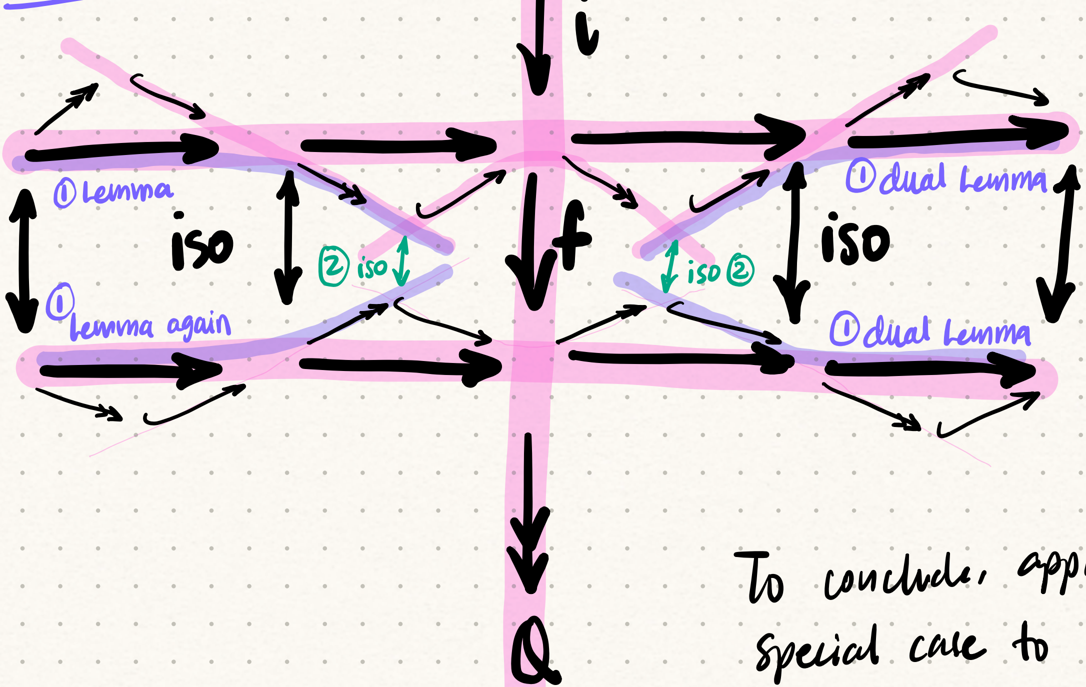


Thm: functoriality of cok

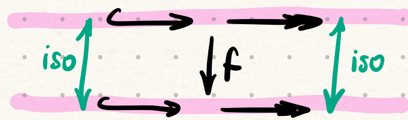


Cor:





To conclude, apply Five Lemma special case to



QED.

Balmer (Liu notes Lem 1.4.8)

Thm "Cartesian $\iff$" (extending "Cartesian $\Rightarrow$" Prop. above)

Thm($\Rightarrow$)$^+$:

Given
$$\begin{tikzcd} & \bullet \arrow[d, "f"] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$
$\rightsquigarrow$ pullback

Then,

$$\begin{tikzcd} & K = \ker(f) \arrow[dl, dashed, "\exists!"'] \arrow[d, "\ker(f)"] \\ \bullet \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner"] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ Q' \arrow[r, hook, "c"] & Q \end{tikzcd}$$

mono!

canonical map (see Functoriality of cok on prev. slide)

Pf:

$$\begin{tikzcd} & \bullet \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner"] & \bullet \arrow[d, "f"] \\ & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ K_{Q'Q} \arrow[r, hook] & Q' \arrow[r] & Q \end{tikzcd}$$

①

Balmer (Liu notes Lem 1.4.8)

Thm "Cartesian $\Leftrightarrow$" (extending "Cartesian $\Rightarrow$" Prop. above)

Thm($\Rightarrow$)$^+$:

Given $\big\downarrow f$ $\rightsquigarrow$ pullback

Then, $\exists!$

$$\begin{tikzcd} & K = \mathrm{Ker}(f) \arrow[d, "\mathrm{ker}(f)"] \arrow[dl, dashed, "\exists!"'] \\ \cdot \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner"] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r] \arrow[d] & \cdot \arrow[d] \\ Q' \arrow[r, hook] & Q \end{tikzcd}$$

mono!

canonical map (see Functoriality of cok on prev. slide)

Cor: (1.4.11)

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

biCartesian $\Leftrightarrow$ 2 vertical arrows $\downarrow, \downarrow$ same ker & coker

$\Leftrightarrow$ 2 horiz $\rightrightarrows$ have same ker & cok.

Pf:

$$\begin{tikzcd} & \cdot \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner\ \equiv"] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner\ (2)\ \equiv"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\equiv"] & \cdot \arrow[d] \\ K_{Q'Q} \arrow[r, hook, "i"'] & Q' \arrow[r] & Q \end{tikzcd}$$

(1)

bonus epi & mono from Cor "Cart $\Rightarrow$" Prop.

$\Rightarrow$

$$\begin{tikzcd} & \cdot \arrow[rr] \arrow[dd, "f'"'] \arrow[ddrr, phantom, "\lrcorner\ \equiv"] & & \cdot \arrow[d] \arrow[dd, bend left, "f"] \\ & & & I_f \arrow[d, hook] \\ \cdot \arrow[r, hook] \arrow[d] \arrow[uurrr, bend left, red, "(4)\ \exists!"] \arrow[dr, phantom, "\lrcorner\ \equiv"] & \cdot \arrow[rr] \arrow[d] \arrow[drr, phantom, "\equiv"] & & \cdot \arrow[d] \\ K_{Q'Q} \arrow[r, hook, "i"'] & Q' \arrow[rr] & & Q \end{tikzcd}$$

(3) (2) (1)

purple exactness from Lemma from Five Lem slides.

(1) $\Rightarrow$ (2) = (3) $\Rightarrow$ (4)

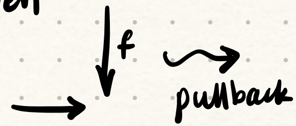
UP ker

Balmer (Liu notes Lem 1.4.8)

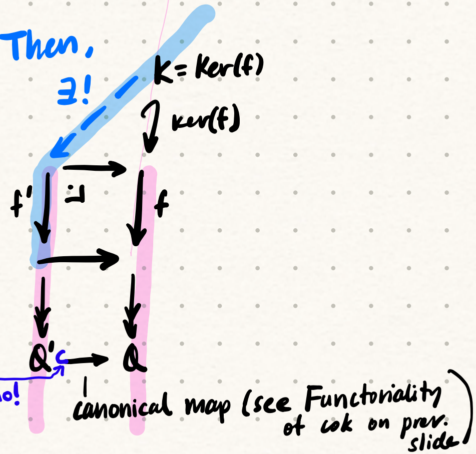
Thm "Cartesian ⇔" (extending "Cartesian ⇒" Prop. above)

Thm(⇒) $^{+}$:

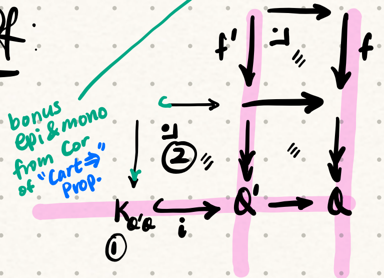
Given



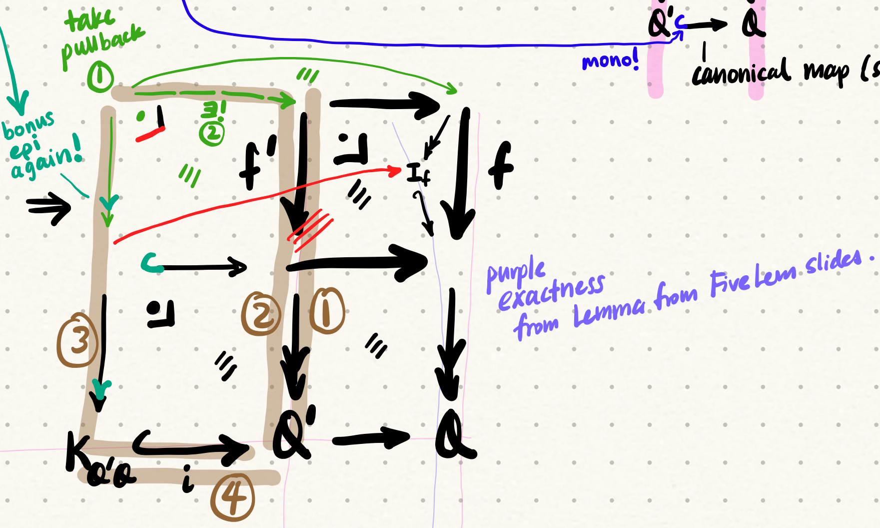
Then,



Pf:



⇒



① ⇒ ② = ③ ⇒ ④
up epi!

Thus $K \hookrightarrow Q' \hookrightarrow Q$, 0

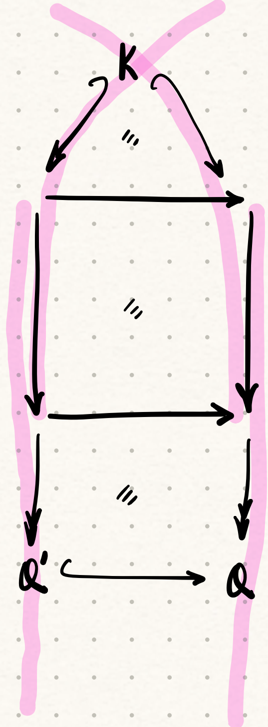
⇒ mono!

a previous prop

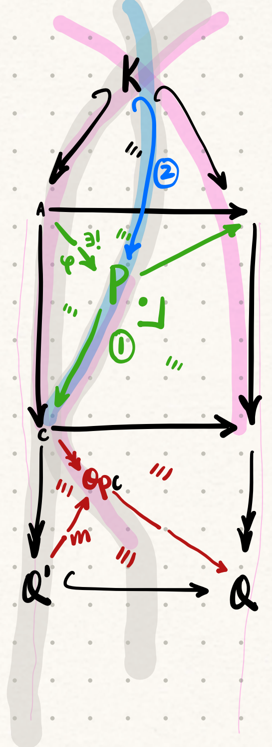
QED ((⇒) dir of Cart⇔ Thm)

Thm ($\Leftarrow$):

Start w/ :



$\Rightarrow$



① take pullback

② "Cart $\Rightarrow$" Prop (1.2.7) says kernels of 2 cols of ⌟ are $\cong$.

③ take cok $\begin{matrix} P \\ \downarrow \\ C \end{matrix}$. Dashed red arrows and $=$ follow from functoriality of cok.

But
$$\begin{tikzcd} C \arrow[d] \arrow[dr] & \\ Q' \arrow[r, dashed, "m"'] & Q_{pc} \end{tikzcd}$$
(prev. props) implies m is epi

and
$$\begin{tikzcd} & Q_{pc} \arrow[dr] & \\ Q' \arrow[ur, dashed, "m"] \arrow[rr, hook] & & Q \end{tikzcd}$$
implies m is mono.

By prev. propositions, mono $\Rightarrow$ ker$=0$
epi $\Rightarrow$ Cok$=0$
$\Downarrow$
iso.

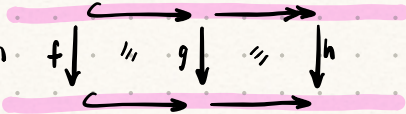
To conclude $A \xrightarrow{\varphi} P$ is iso, apply Five Lemma.

Balmer (Liu note Lem 1.4.12)

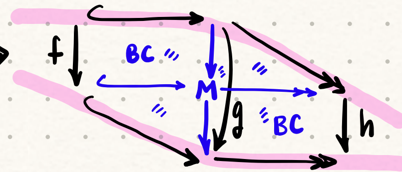
2 BC / hexagon thm : Given (morphism b/t 2 SES)

bicartesian

(SES morphism factorization)

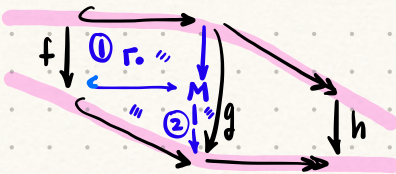


$\rightsquigarrow$ $\exists!$:



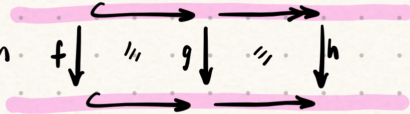
Pf: ① take pushout of top left square. ② by UP ⌜.

(prev. Prop ⌜ of mono is mono)

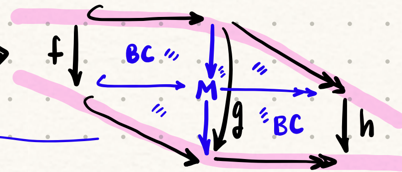


Balmer (Liu note Lem 1.4.12)

2 BC / hexagon Thm: Given (morphism b/t 2 SES)
bicartesian
(SES morphism factorization)

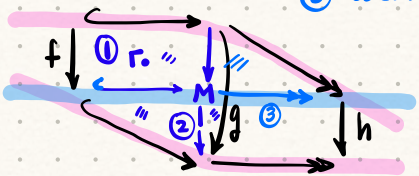


⇝ ∃! : unique up to unique iso, b/c ⌐ and/or ≅ are unique up to unique iso

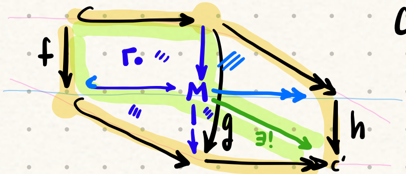


Pf: ① take pushout of top left square. ② by UP ⌐ (prev. Prop ⌐ of mono is mono)

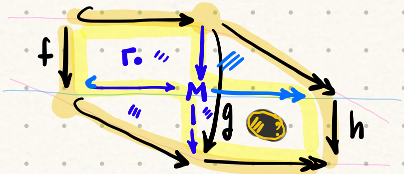
③ coCartesian ⇝ (1.2.7) same cokernels (2 horiz arrs ↪ have...)



But now 2 horiz arrs ↪ have same cok & ker (= 0)
so by "Cart ⇔" Thm, BC ✓



Consider the 2 yellow morphisms.
By UP ⌐, get unique! green $M \to C'$
so that 2 green highlighted are equal (and =), "commutativity prop"



all 4 yellow highlighted paths are equal.

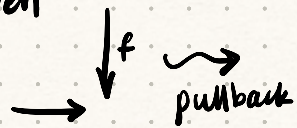
UP ⌐ Uniqueness of $M \to C'$ satisfying "commutativity prop" implied ✓ commutative!

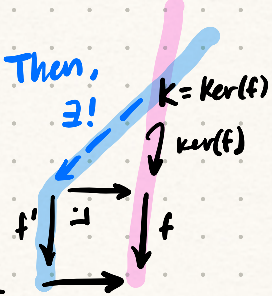
To conclude, look at bottom right comm. square. 2 horiz. ↠ same ker, same cok = 0. so by "Cart ⇔" Thm, BC ✓

Liu's notes on
Balmer 212A

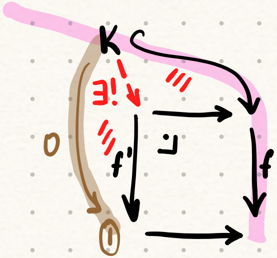
Junk

Prop: Given





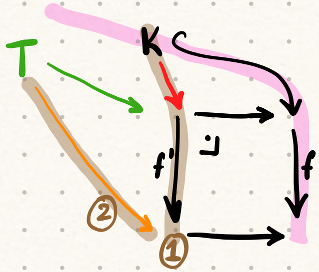
Pf:

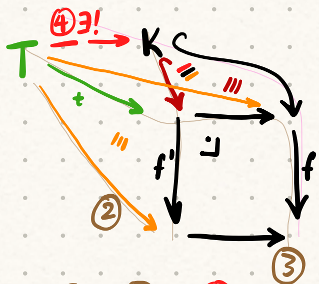


by UP ⌟

Consider test object

T s.t. :





② ⇒ ③ ⇒ ④
UP
ker

=
equiv
to
2 yellow
highlight
equal