Calc III and Beyond

Daniel K. Rui - June 6, 2025

Abstract

For (my) Calc III students. Regarding the later admittedly extremely cryptic sketches, ask me (or any other sufficiently well-versed grad student) to explain in more detail.

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1 Calc III Final Exercises

One last conceptual exercise (sorry, I couldn't resist 😟): read `https://courses.lumenlearning.com/calculus3/chapter/flux-form-of-greens-theorem/`, and fit it into the framework I presented in Week 9: define ${}^1\omega_F$ (i.e. if I feed it an oriented 1-dim. segment $\nearrow$ with midpoint $\bullet$, what should the number ${}^1\omega_F(\nearrow)$ be defined as, in terms of $\nearrow$ and $\bullet$?), then define ${}^{2,1}\omega_F(\vec{\square})$ by using the "sum over boundary" operation on ${}^1\omega_F$. (Hint: for small oriented parallelograms $\vec{\square}$ with center $\bullet$, ${}^{2,1}\omega_F(\vec{\square})$ should be $\approx \operatorname{div} F(\bullet)[dx\,dy](\vec{\square})$).

From here, derive an FTC (exactly the way I did it; using the principle that contributions from the boundary between 2 adjacent parallelograms should cancel).

${}^1\omega_F(\nearrow) := F(\bullet) \cdot \text{normal}(\nearrow) = F(\bullet) \cdot \hat{n}\, ds(\nearrow)$

`https://www.youtube.com/watch?v=np2TVlWoSOQ&ab_channel=JoeBreenMath` Joe Breen [legendary UCLA math grad student from some years ago] 3 hour 32B review session with tons of good problems. There is even a 3 hour part 2 `https://www.youtube.com/watch?v=QdTAJUQujkO&ab_channel=JoeBreenMath`

1. Let C be the oriented closed curve consisting of straight-line segments from $(0,0,0)$ to $(1,1,1)$ to $(0,2,3)$ to $(-1,1,2)$ and the back to $(0,0,0)$. Let $\vec{F}(x,y,z) := \langle -yx, x^2, e^{z^3}\rangle$. Compute $\int_C \vec{F}\cdot d\vec{r}$. What tools are there? Direct eval? [e^{z^3} has bad vibes...] FTC? [When can you apply FTC?] Stokes' theorem? [What surface? 3 points determine a plane...]

2. Let $\vec{F}(x,y,z) := \frac{1}{x^2+y^2+z^2}\langle x,y,z\rangle$ be a vector field on $\mathbb{R}^3 \smallsetminus \{(0,0,0)\}$. Does $\vec{F}$ have a (vector) potential, i.e. is there a vector field $\vec{A}$ (on $\mathbb{R}^3 \smallsetminus \{(0,0,0)\}$) s.t. $\vec{F} = \operatorname{curl}\vec{G}$? What do you check first [a "necessary condition"] for these potential problems? **IF** a vector field *does* have a potential, what does FTC/Green/Stokes say?

3. Let $\vec{F}(x,y) := \frac{1}{x^2+y^2}(-y,x)$ be a vector field on $\mathbb{R}^2 \smallsetminus \{(0,0)\}$. Does $\vec{F}$ have a (scalar) potential, i.e. is there a scalar function f (on $\mathbb{R}^2 \smallsetminus \{(0,0)\}$) s.t. $\vec{F} = \nabla f$? What do you check first [a "necessary condition"] for these potential problems? **IF** a vector field *does* have a potential, what does FTC/Green/Stokes say?

4. Let $\vec{F}(x,y,z) := \frac{1}{(x^2+y^2+z^2)^{3/2}}\langle x,y,z\rangle$ be a vector field on $\mathbb{R}^3 \smallsetminus \{(0,0,0)\}$. Does $\vec{F}$ have a (vector) potential, i.e. is there a vector field $\vec{A}$ (on $\mathbb{R}^3 \smallsetminus \{(0,0,0)\}$) s.t. $\vec{F} = \operatorname{curl}\vec{G}$? What do you check first [a "necessary condition"] for these potential problems? **IF** a vector field *does* have a potential, what does FTC/Green/Stokes say?

5. Let C_1 be the circle parameterized by $\vec{r}_1(t) = \langle \cos(\frac{e^t}{1+t^2}\arctan t), \sin(\frac{e^t}{1+t^2}\arctan t), 0\rangle$ (over a time interval s.t. the parameterization makes 1 full loop) and C_2 be the circle parameterized by $\vec{r}_2(t) = \langle \sin t, \cos t, 0\rangle$. I have a secret vector field $\vec{F}$ (defined and smooth at every point of $\mathbb{R}^3$!) s.t. $\operatorname{curl} F(x,y,z) = \langle 0,0,z^2\rangle$ whenever (x,y,z) are on the cylinder $x^2+y^2=1$. [Note that the divergence of the vector field $G(x,y,z) = (0,0,z^2)$ is not zero, so G can't be the curl of another vector field... and yet... How can it be that I am still telling the truth? Challenge problem: find a $\vec{F}$ satisfying the above condition.] Suppose I tell you that $\int_{C_2}\vec{F}\cdot d\vec{r} = 5$. What is $\int_{C_1}\vec{F}\cdot d\vec{r}$?

6. Let the curve C be given by $y = \frac{1}{e^x+\ln x}$ for $1 \leqslant x \leqslant e$ oriented from left to right. Let $\vec{F}(x,y) = \langle 3x^2+y^2, 2xy\rangle$. Evaluate $\int_C \vec{F}\cdot d\vec{r}$. What tools are there? Direct eval? FTC/Green/Stokes?

https://www.cliffsnotes.com/study-notes/19664269

1. Consider the integral

$$\int_0^{\sqrt{2}/2}\int_x^{\sqrt{1-x^2}} dy\,dx + \int_{-\sqrt{2}/2}^{\sqrt{2}/2}\int_{|y|}^{\sqrt{1-y^2}} dx\,dy.$$

Use polar coordinates to combine the integrals into a single double integral, and evaluate it.

2. Consider the solid $\mathscr{W}$ bounded below by the xy-plane, on the sides by the sphere $\rho = 2$, and above by the cone $\phi = \frac{\pi}{3}$. Find the spherical coordinate limits for the integral that calculates the volume of the region $\mathscr{W}$, and evaluate the integral.

https://www.math.ucla.edu/~colinni/32B_final_review_problems.pdf (see in particular the true/false questions!)

Problem 2: Compute

$$\iint_S \left\langle -2xe^{z^2}z, y\sin(z), e^{z^2}+\cos(z)\right\rangle \cdot d\vec{S}$$

where S is the upper half of the sphere of radius 3 centered at the origin $(0,0,0)\in\mathbb{R}^3$.What tools are there? Direct eval? FTC/Green/Stokes? [What would you need to apply Stokes? Can you "manifest" that?]

A very similar problem to the previous one: Problem $\tilde{2}$: Compute

$$\iint_S \left\langle -2xe^{z^2}, y(\sin z)^2, e^{z^2} + \cos(z) \right\rangle \cdot \, d\vec{S}$$

where S is the upper half of the sphere of radius 3 centered at the origin $(0, 0, 0) \in \mathbb{R}^3$.What tools are there? Direct eval? FTC/Green/Stokes? [What would you need to apply Stokes? Can you "manifest" that?]

https://www.math.ucla.edu/~mjandr/Math32B/final.pdf I think these problems are quite good!

2 What's next, after Calc III?

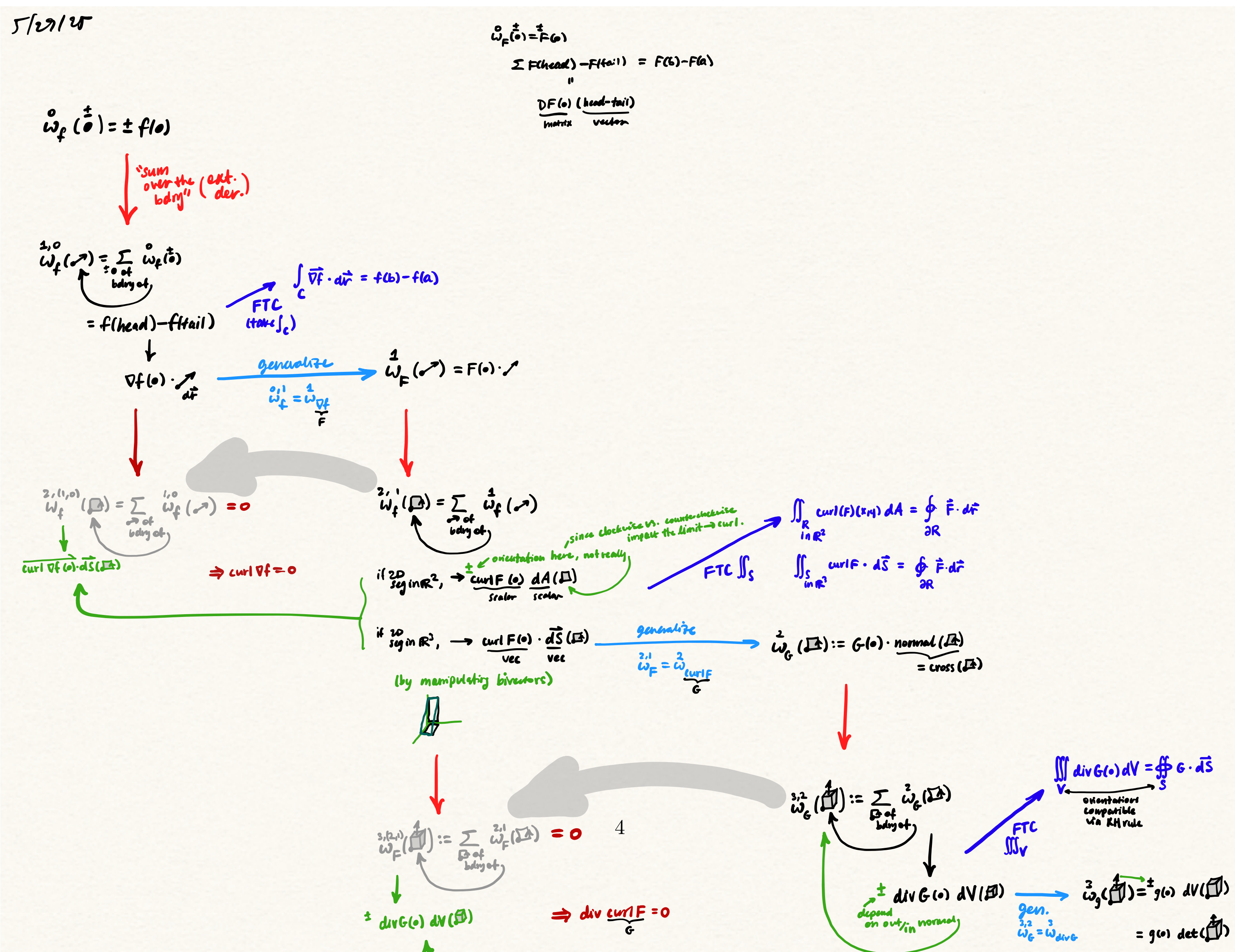
2.1 Previously

In Calc III, you learned about integrals using the following notations:

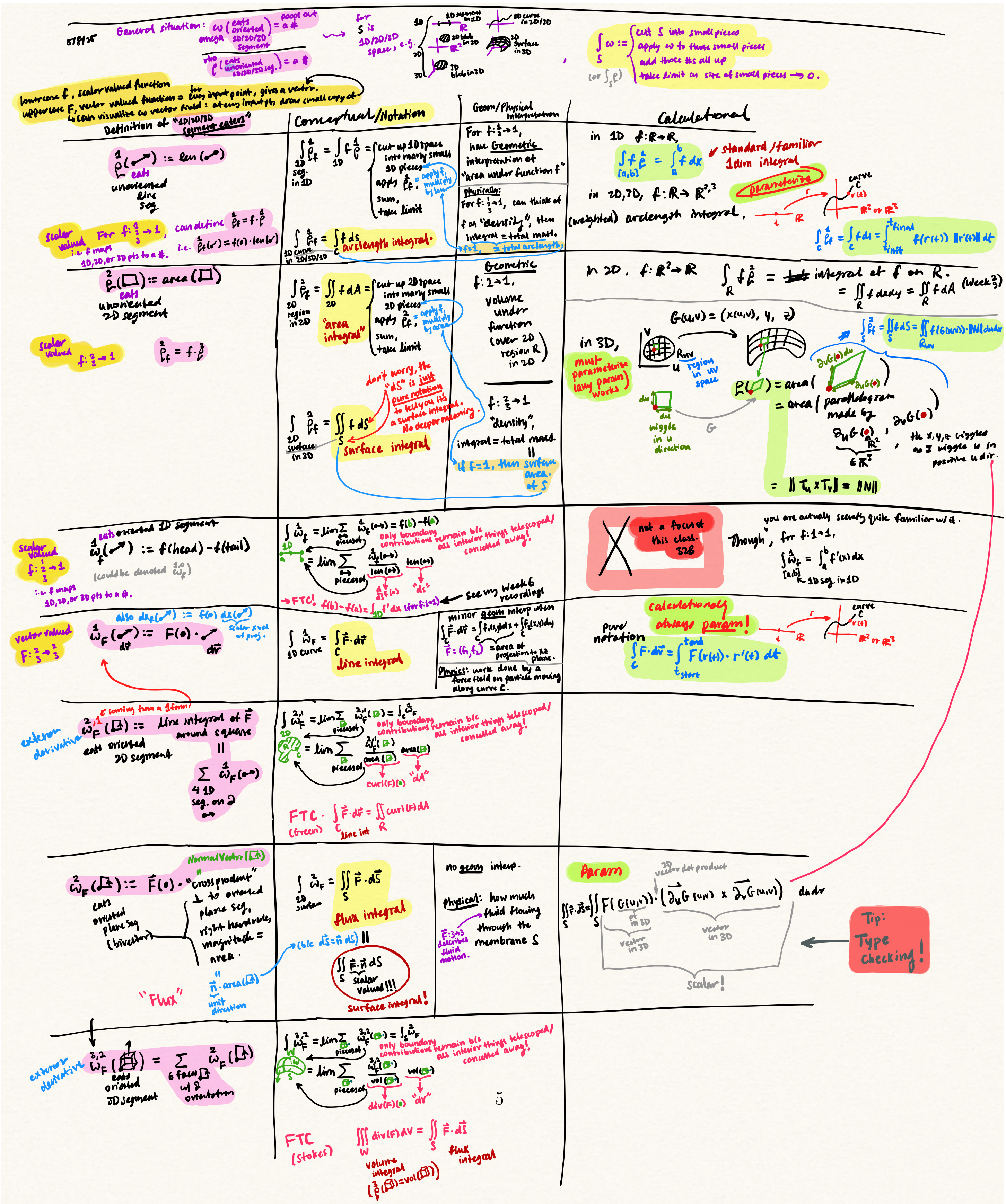
- dx, dy, dz
- ds (arclength integrals)
- $\vec{dr}$ (line integrals)
- dS (surface integrals)
- $\vec{dS}$ (flux integrals)
- dA
- $dx\,dy$, $dy\,dz$, $dx\,dz$
- dV
- $dx\,dy\,dz$.

Not just notation, but have meaning. All of them eat 1/2/3-dimensional segments and output numbers (e.g. eat line segments, plane segments like triangles or parallelograms, or space segments like cuboids or paralleopipeds or tetrahedra), but some (I denote ϱ) eat *unoriented* such segments, and some (I denote ω) eat *oriented* such segments.

Here's a single flow chart/pipeline building up all of the oriented "segment eaters" from a single seed (evaluation of a function at an *oriented* point), using only 2 operations: "sum over boundary" and "generalize", and discussing how to derive "FTC"-like theorems (FTC, Green's theorem, Stokes' theorem), and also the identities $\operatorname{curl}\operatorname{grad} f = 0, \operatorname{div}\operatorname{curl} F = 0$.



Here's a table summarizing all the "segment eaters":



2.1.1 Defining the Integral

For a "1/2/3-dim. segment eater" ϱ or ω as above (eating segments of a certain n-dimensional space ["manifold"] M), we can define the integral $\int_M \omega$ (similarly $\int_M \varrho$) as the result of the following recipe:

$$\int_M \omega := \begin{cases} \text{cut } M \text{ into small } n\text{-dim. (oriented) segments} \\ \text{apply } \omega \text{ to those segments, getting numbers out} \\ \text{sum those numbers} \\ \text{take the limit as the size of the segments} \to 0 \end{cases}$$

This is an intuitive/conceptual "definition", but it is not rigorous because I haven't justified why this limit exists! However, from this intuitive/conceptual "definition", I derived (again only from mostly intuitive/conceptual grounds) a calculational formula, which is completely rigorous. However, to use that formula, one must **choose** a parameterization of M! I did not show that it is independent of the parameterization, though the textbook does.

You can imagine that substantial work has gone into making my above intuitive/conceptual "definition" rigorous (which is important due to subtle/surprising pathologies already apparent in say Schwarz lanterns — see also Mathologer video), and also developing a more streamlined language in which all these rather technical things can be formulated more cleanly.

An example of this idea of developing streamlined mathematical language: in Calc III, we study some vectors, matrices, linear transformations, planes, dot products, etc. These topics may be scattered haphazardly in your mental landscape right now, but actually underlying it all is a rich coherent narrative that people discovered and organized into the theory of *(abstract) linear algebra*. See 3b1b's Linear Algebra video series for a visual introduction.

Another example of why mathematicians want to develop this more streamlined language is the change of variables formula from Calc III. MO Tim Gowers quote:

> ""There are a number of facts in multivariable calculus that are obvious but hard to prove. For instance, the change-of-variables formula in a multiple integral is very easy to justify heuristically by talking about little parallelepipeds but troublesome (as I discovered to my cost in a course I once gave) to justify rigorously. [...]
>
> I'd be quite glad to be told that this answer was wrong. If anyone knows of a link to an exposition of these results [...] that does proper justice to their intuitive obviousness, I'd be very pleased to hear about it." - Tim Gowers 2011
>
> "I'm afraid you're right. The change of variables formula for multiple integrals is a notorious "Is it really this hard?" moment in mathematical exposition." – Pete L. Clark 2011"

More modern (past 50 years style) proofs:
Analysts' way: `https://actamath.savbb.sk/pdf/acta1906.pdf`
Algebraists' way: *Global Calculus* by Ramanan, Chapter 3

2.2 Journey to the Orient(ed): Bivectors, Linearity

Holy shit we can add/scale parallelograms?! $\rightsquigarrow$ Holy shit we can add/scale surfaces?! Add, scale... these are vector operations! Holy shit we can do linear algebra on surfaces?!?!

From Zero to Geo for excellent visual introduction to bivectors. The weird way we add them, is forced upon us because that's exactly the effect we need when we work with ${}^2\omega_F(\vec{\square})$, to show convergence to $\text{curl}(F)(\bullet)\, d\vec{S}(\vec{\square})$ (where $\bullet$ refers to say the center of the oriented parallelogram $\vec{\square}$ in $\mathbb{R}^3$), and also to derive exterior derivative formula.

2.3 Exterior Derivative

$\text{curl}\,\text{grad}\, f = 0, \text{div}\,\text{curl}\, F = 0 \rightsquigarrow d^2 = 0$, *explained/caused by* $\partial^2 = 0$.

2.4 What Antiderivatives are Possible?

In $\mathbb{R} \smallsetminus \{0\}$, anti-derivatives of $\equiv 0$ are $C_1 \cdot 1_{x<0}$ and $C_2 \cdot 1_{x>0}$. That's why the antiderivatives of $\frac{1}{x}$ is not actually $\ln|x| + C$, but rather $\ln|x| + C_1 \cdot 1_{x<0} + C_2 \cdot 1_{x>0}$.

closed = "d is 0"

exact = "d of something"

Poincare lemma (convex, or star-shaped domains): https://en.wikipedia.org/wiki/Poincar%C3%A9_lemma#Proof_in_the_two-dimensional_case has proof for 2-dim. case that Calc III students can understand

So calculus can prove a topological theorem about non-homotopic paths!

$\mathbb{R}^2 \smallsetminus \{0\}$

The differential form $\frac{y\,dx - x\,dy}{x^2+y^2}$ is closed (its derivative is 0) but not exact (is not the derivative $df = \partial_x f\, dx + \partial_y f\, dy$ of some function f). It is the topology of $\mathbb{C} \smallsetminus \{0\} \cong \mathbb{R}^2 \smallsetminus \{0\}$ that obstructs the gluing from local to global.

2.4.1 More Local to Global

The simplest substantive Example of a line bundle is the Möbius band. First, imagine (or better, physically hold) a rectangular strip of paper $[0, 2\pi] \times (-1, 1)$. Any (continuous, smooth) function $f : [0, 2\pi] \to (-1, 1)$ can be graphed as a (continuous, smooth) curve on this strip of paper. Now imagine (or physically do) bending the strip so that the ends meet, i.e. $\{0\} \times (-1, 1)$ becomes identified with $\{1\} \times (-1, 1)$, and we get a cylinder (see the picture here). Then, the graph of a continuous (resp. smooth) function $f : [0, 2\pi] \times (-1, 1)$ becomes a continuous (resp. smooth) curve on the cylinder iff $f(0) = f(2\pi)$ (resp. all one-sided derivatives match). We can think of these as graphs of functions $f : S^1 \to (-1, 1)$ (S^1 being the circle).

Alternatively, we could *twist* the strip before attaching/identifying the ends, i.e. $(0, y) \in \{0\} \times (-1, 1)$ becomes identified with $(2\pi, -y) \in \{2\pi\} \times (-1, 1)$ (see this MSE post for rigorous formulas). This produces a Möbius band, which I'll call M. There are a bajillion videos online if you need help visualizing (or, you could play with paper strips yourself). Then, the graph of a continuous (resp. smooth) function $f : [0, 2\pi] \times (-1, 1)$ becomes a continuous (resp. smooth) curve on M iff $f(0) = -f(2\pi)$ (resp. all one-sided derivatives are the negation of each other $f^{(k)}(0+) = -f^{(k)}(2\pi-)$). We can think of these as graphs of "functions on S^1 in M".

I illustrated one example of such a "function", in red on the picture on the below right. Notice that by the intermediate value theorem, in fact all "functions on S^1 in M" must have a zero (i.e. hit the "base" blue circle S^1). So somehow, the global geometry of M (the "twist") puts a very strong restriction on the behavior of "functions on S^1 in M": they must vanish somewhere! Note that the cylinder C has no such restriction. Note also that locally (on little "snippets" corresponding to some small $U \subsetneq S^1$), both C, M are identical to just a rectangle $U \times (-1, 1)$. In particular, locally, functions have no restriction on their behavior. This I think is the **fundamental/foundational observation behind line bundles**: locally, you can have whatever functions you want, but as you glue to get "functions" on a larger and larger domain, the global geometry then starts obstructing you.

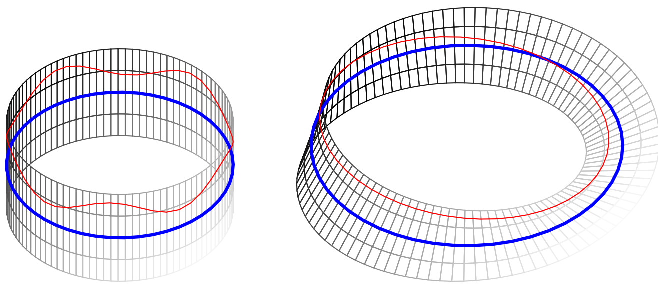


Figure 1: Line bundles on S^1 (graphics sourced mainly from this TEX.SE post)

The cylinder C and Möbius band M (in the graphic above, M, C are resp. the images of the two $[0, 2\pi] \times (-1, 1) \to \mathbb{R}^3$ maps $(x, l) \mapsto (\cos x, \sin x, 0) + \frac{1}{4}l \cdot (\cos(\frac{x}{2}) \cos x, \cos(\frac{x}{2}) \sin x, \sin(\frac{x}{2}))$ and $(x, l) \mapsto (\cos x, \sin x, l)$) are both examples of a line bundle on S^1

2.5 Integral Pairing and de Rham's Theorem

https://math.stackexchange.com/questions/1886437/integration-of-a-cohomology-class-over-a-homology-c

https://en.wikipedia.org/wiki/De_Rham_theorem some massive-brain person went through and wrote that the derived ∞-category is the correct setting in which to argue de Rham's theorem. The proof outlined brilliant showcases the power and ideal of modern (abstract algebraic) mathematics; I wonder who wrote it...

de Rham's theorem gives a beautiful bridge between Calc III/cohomology, with homology.

2.6 de Rham: geometry, or ... algebra? Philosophy of different proofs

https://math.uchicago.edu/~may/REU2018/REUPapers/Chaiyachakorn.pdf DE RHAM'S THEOREM, TWICE

NICK CHAIYACHAKORN Abstract. We give two proofs of de Rham's theorem, showing that de Rham cohomology and singular homology are isomorphic on smooth manifolds. The first involves the Eilenberg-Steenrod axioms for homology and a proof technique called the Mayer-Vietoris argument on manifolds; the second involves sheaf cohomology. We introduce homological algebra and sheaf theory as required.

Pursuing further this bridge between (singular) homology and de Rham cohomology given by integration, one can prove de Rham's theorem. This is the proof one would see in a graduate-level geometry/topology course. Abstract algebra (in particular homological algebra) already makes its presence known/unavoidable, but still there is a deep geometric flavor.

The surprising fact that has only been realized more recently is that actually, de Rham's theorem in some sense doesn't have that much to do with geometry at all. Indeed, as we've developed abstract algebra more and more, we have seen that de Rham's theorem actually follows from almost pure algebra. Though perhaps the philosophical lesson is not "de Rham's theorem is not geometry, but rather pure algebra"; but instead "de Rham's theorem is clearly geometry, yet deep down somehow it is pure algebra, and so geometry deep down must also be pure algebra".

2.6.1 Analogies to understand mathematicians' philosophical motivations

An analogy: why do mathematicians care about finding "better" proofs/looking from a different perspectives? von Neumann fly problem. From the distance perspective, it's a infinite series problem fit for a 31B Calc II student. From the time perspective, it's a middle school exercise.

Another analogy more fitting to the derived ∞-category proof of de Rham: have a short intuitive argument, but just need to formalize the words correctly so that that short intuitive argument actually becomes logical/airtight. "Windows to higher realms." [See also §2.10] `https://en.wikipedia.org/wiki/Umbral_calculus`, searching 'physicists calculus' $e^x - \int e^x = 1 \implies e^x = (1-\int)^{-1}0 = 1 + x + \frac{1}{2!}x^2 + \ldots$ which is made rigorous by operators on Banach spaces/Banach FPT. Or the famous series manipulation $1 + 2 + 3 + \ldots = -\frac{1}{12}$, that is made rigorous by divergent sum techniques and complex analysis.

Distribution theory in analysis: e.g. Clairut's theorem is always true, not for standard notion of derivative, but for weak/distributional notion of derivative.

Another analogy: suppose that you have just mastered addition, and suddenly I give you the problem: if I make a grid of 179 rows and 53 columns, and fill each cell with a stone, then how many stones are there in total? You can of course solve this problem: simply add 179 to itself 53 times (which you know how to do, since you know addition). However, the very nature of the problem *compels* another solution: the development of the theory of multiplication. New patterns emerge, like the connection between multiplying by 10 and shifting to the left. This new pattern leads us to the idea of decomposing 179 as $100 + 70 + 9$, and multiplying each piece with 53 separately (reducing the problem to multiplying a 2-digit number with just 1-digit numbers, and then shifting to the left).

It's not that addition didn't give us the right answer; of course it did. In some sense, we aren't doing anything new. But of course, knowing multiplication as you do, I think you would agree with me that developing the theory of multiplication, is clearly the right thing to do for this problem. Furthermore, after developing this new language, we are led to questions we probably would not have considered before, like the notion of prime numbers, factoring, etc.

By studying old problems that we technically can solve with old methods, we slowly realize that the old problems are *compelling* us to do something new, just like how the above stones-in-a-grid problem compelled us to go beyond addition to multiplication.

2.7 Philosophy: a Space and its Functions

Torus, punctured torus?

This philosophy/theme pervades so much of modern math: sheaf theory/cohomology, algebraic geometry, Yoneda lemma in category theory

2.8 Calculus $\rightsquigarrow$ Cohomology $\rightsquigarrow$ Homology

Simplicial homology is a good place to get started learning more (about topology, and I think more importantly about this whole machinery of homology). The definitions of say the boundary differential are famously confusing (with all the $\pm$ signs...), but that's what so important about the Calc III $\rightsquigarrow$ (de Rham) cohomology $\rightsquigarrow$ (singular) homology pipeline: just following this pipeline, the universe leads us by the hand to a new vista (namely, the idea the we should really care about orientations of segments, and induced orientation on the boundary so that the boundary between 2 adjacent segments should cancel; relatedly, the idea $\partial^2 = 0$ — I discuss this more below) that we would have been very very unlikely to invent unprompted by ourselves, but is relatively easy to explain once we take-for-granted a handful of "miraculous" definitions.

Here is what I think is the philosophically correct way to think about "orientation" in this business: for 1/2/3-dim. segments (think line segments, plane segments = parallelograms, and space segments = parallelopipeds), we have 2 natural options: forward/backward, clockwise/counterclockwise, inward/outward normal (resp.). We call all of these orientations since we as humans have an intuitive sense of that word and it matches well with these 2-options per dimension.

However, what Calc III reveals to us is that making a Selection of 1-out-of-the-2-options for a 1/2/3-dim. segment induces a Selection on its boundary, in such a way that neighboring segments sharing some boundary have opposite induced Selection for that shared boundary. And this enables the cancellation/telescoping behind Stokes' theorem/FTC-phenomena. Furthermore, the cohesive story of Stokes' theorem involves extrapolating the "orientation" backwards to 0-dim. points, which are "oriented" by assigning them either $+$ or $-$.

This is the universe telling us that actually, our mushy human intuitive sense of orientation, gives rise to a cold hard fact: when we allow ourselves something as seemingly minor as a binary freedom/choice to Select one of $\pm$ for segments, there is a way to induce such a Selection of $\pm$ on the boundaries, in such a way that advantageous cancellation happens. This is how we should think of orientation. Indeed, not only can I say "there exists" a way, but I can actually find a "formula" for it, by ordering the vertices:see MSE answer.

The thought pipeline goes:

Observing 2 naturally appearing symmetrical choices when dealing with 1/2/3-dim. segments and intuitively associating that with the mushy/vague human idea of orientation

$\rightsquigarrow$ mathematical discovery on Selecting 1-out-of-2-choices and advantageous cancellation

$\rightsquigarrow$ realization that this is the mathematical/philosophical heart of the naturally occurring 2-choices we kept Observing at the beginning — so not really having anything to do with the mushy/vague human intuition of orientation, *especially* in higher dimensions where our intuition is completely dead; but we still continue to call it "orientation" anyways because that's a nice word.

Some sources to learn about simplicial homology:

- `https://www.jeremykun.com/2013/04/03/homology-theory-a-primer/` blog series
- `https://web.archive.org/web/20200917230750/https://eric-bunch.github.io/blog/calculating_homology_of_simplicial_complex` blog post
- `https://adamgyenge.gitlab.io/teaching/info3/2025/lec10.pdf` slides
- `https://abouthydrology.blogspot.com/2017/09/a-smooth-introduction-to-some-algebraic.html` blog series (+ videos) for persistent homology (topological data analysis)
- `https://algebrology.github.io/simplicial-complexes-and-boundary-maps/` nice (pretty LaTeX) blog post
- `https://www.cs.ubc.ca/~jf/courses/531F.S2025/Handouts/simplicial_intro2025.pdf` big long notes pdf, self contained

- https://www.fields.utoronto.ca/programs/scientific/04-05/data_sets/parent.pdf sketch/summary/overv not self contained but good to get a sense of bearings (one can more easily determine the overarching structure from a sketch than from something already inlaid with all the intricate details)
- https://pageperso.lis-lab.fr/~shantanu.das/imdseminar/papers/talkYannSitu.pdf nice slides
- https://www.math.uci.edu/~mathcircle/materials/MCsimplex.pdf VERY NICE worksheet to learn simplicial complexes (but no homology)
- https://people.reed.edu/~davidp/411/handouts/simplicial.pdf worksheet for homology, but doesn't introduce the linear algebra language at the beginning
- https://youtu.be/-bM8Gg6bOJo?si=fOeXXIwzunKVTTgP&t=648 nice video lecture on simplicial homology, mostly self-contained in 1st half; 2nd half you have to know what kernel, image, quotient are.

2.9 Remarks for the Unoriented

Measure theory, geometric measure theory. Modern breakthroughs, $\mathbb{R}^3$ Kakeya

2.10 Further Windows to Higher Realms

Above in a paragraph, I gave some examples of windows to higher realms that one may encounter in a "standard" mathematical journey. Here are more:

- $\frac{1}{1-x} = 1 + x + x^2 + \dots$ what if you plug in $x = 2$? In some sense it makes sense... $\rightsquigarrow$ p-adic numbers
- $\frac{1}{1+x^2}$ smooth, beautiful function using most basic operations... but why its Taylor series refuse to converge past radius 1? $\rightsquigarrow$ complex analysis, cf. Hadamard quote
-

See also https://mathoverflow.net/questions/115032/non-rigorous-reasoning-in-rigorous-mathematics https://alexanderpruss.blogspot.com/2009/10/surprising-effectiveness-of-non.html October 2, 2009 The surprising effectiveness of non-rigorous mathematics in physics