290K W24 LACE EXPANSION PART 1 NOTES

1. Cast of Characters for SRW and SAW

Main resource: Gordon Slade's book *The Lace Expansion and its Applications (2005)* https://personal.math.ubc.ca/~slade/sf.pdf

Heavy bombardment of notation incoming!:

- Today, we will use $\Omega := \{x \in \mathbb{Z}^d : \|x\|_1 = 1\}$, the *nearest neighbor model*. In $\mathbb{Z}^2$, this is just the 4 squares directly left of, right of, above, and below the origin (a cross shape missing the origin). $|\Omega| = 2d$

  The reason such notation is introduced for such a trivial object is because people also consider the *spread out model* $\Omega_L := \{x \in \mathbb{Z}^d : 0 < \|x\|_\infty \leq L\}$ for L some large fixed constant (i.e. a $(2L+1) \times (2L+1)$ axes-parallel square centered at the origin, with the origin removed).

  Or in fact more generally, any finite set $\Omega \subseteq \mathbb{Z}^d$ invariant under the symmetry group of $\mathbb{Z}^d$, namely permutation of coordinates or replacement of any coordinate x_i by its negative $-x_i$.

  Like I said for today, Ω is just going to be the nearest neighbor (NN) model, but in fact I will mention a phenomenon of *universality* later.

- We will use ω to denote a *n-step walk taking steps in* Ω, i.e. $\omega : \{0, \dots, n\} \to \mathbb{Z}^d$ s.t. $\omega(i) - \omega(i-1) \in \Omega$ (for $i \in \{1, \dots, n\}$).
- Let $\mathcal{W}_n(x, y)$ denote the set of (NN) walks from $x, y \in \mathbb{Z}^d$. If we just put in one argument, $\mathcal{W}_n(x) := \mathcal{W}_n(0, x)$.
- We let $c_n^{(0)}(x, y) := |\mathcal{W}_n(x, y)|$ and of course $c_n^{(0)}(x) := c_n^{(0)}(0, x)$.
- Similarly let $\mathcal{S}_n(x, y)$ denote the set of (NN) SAW from $x, y \in \mathbb{Z}^d$; and let $c_n^{(1)}(x, y) := |\mathcal{S}_n(x, y)|$.
- So far, these "c" quantities are all combinatorial, i.e. they literally count things. We will now reinterpret them in the following light: for $\lambda = 0, 1$,

$$c_n^{(\lambda)}(x, y) := \sum_{\omega \in \mathcal{W}_n(x,y)} \underbrace{\prod_{0 \leq s < t \leq n} (1 \overbrace{- \lambda_{(st)} \mathbb{1}_{\omega(s) = \omega(t)}}^{=: U_{st}})}_{=: K^{(\lambda)}[0,n](\omega)}.$$

The $(1 + U_{st})$ factor "penalizes" by factor of $(1 - \lambda)$ for every self intersection.

Every $\omega \in \mathcal{W}_n(x,y)$ is assigned the weight/fre"K"uency/"scaled likelihood" $K^{(\lambda)}[0,n](\omega)$, which one can interpret as: "assuming 'λ-level' of self repulsion (0 is no self repulsion, 1 is completely self avoiding), this is 'frequency' or 'scaled likelihood' of seeing the path ω."

In other words, before, every path in $\mathcal{W}_n(x,y)$ was equally likely, but now, due to it being "harder" to self intersect, self-intersections (especially many self-intersections) are more rarely seen, where $K[0,n](\omega) \in [0,1]$ quantifies how rare (the smaller it is, the more rare the configuration).

Now we have the interpretation that $c_n^{(\lambda)}(x,y)$ is the sum of all weights/frequencies of the (W)SAWs going from x to y in n steps, and

$$\frac{K[0,n](\omega)}{c_n^{(\lambda)}(0,y)}$$

is the genuine likelihood of seeing a specified path ω in $\mathcal{W}_n(0,y)$.

- Now we define the number

$$c_n^{(\lambda)} := \sum_{y\in\mathbb{Z}^d} c_n^{(\lambda)}(0,y)$$

, i.e. summing over all endpoints $y \in \mathbb{Z}^d$ all the frequencies of n-step paths $0 \to y$, so a total frequency count for every n-step path starting at 0.

For example, for SRW i.e. $\lambda = 0$, we have $c_1^{(0)} = |\Omega| = 2d$ and $c_n^{(0)} = |\Omega|^d = (2d)^d$.

We are truly interested in $c_n^{(1)}$, which recall is the total number of n-step SAW starting at 0, i.e. in the previous notation $c_n^{(1)} := \left|\bigcup_{x\in\mathbb{Z}^d} \mathcal{S}_n(0,x)\right|$

- And then the ratio

$$\mathbb{P}_n^{\lambda}(y) := \frac{c_n^{(\lambda)}(y)}{c_n^{\lambda}}$$

is exactly the likelihood of seeing any n-step (W)SAW ending at y.

Taking expectations with respect to this probability measure on $\mathbb{Z}^d$ is denoted with $\mathbb{E}_n^{\lambda}$.

- We know that for all $\lambda \in [0,1]$, the sequence $c_n^{(\lambda)}$ is log-subadditive:

$$c_{n+m} \le c_m c_n$$

because the RHS comes from forgetting the (penalizing!) interactions between the left and right half the walk, and fewer penalities means a higher value.

A simple exercise in real analysis (it has a name — Fekete's lemma!) tells us then that

$$\lim_{n\to\infty} c_n^{1/n}$$

exists; let us call it $\mu^{(\lambda)}$.

For example, $\mu^{(0)} = 2d$.

For the quantity we care about $c_n^{(1)}$, the corresponding limit value $\mu^{(1)}$ is called the *connective constant*. Unknown for $d = 2, 3, 4$. Obviously $\le 2d$. And by looking at all walks using only "right" and "up" moves, $\ge d$.

- Let me rephrase the above limit:

$$\lim_{n\to\infty} c_n^{1/n} = \mu \iff c_n = \underbrace{(1 + o_{n\to\infty}(1))^n}_{?} \cdot \mu^n.$$

We want to understand better the quantity labelled by "?", i.e. we want to understand the quantity c_n/μ^n.

- Here's the clever idea: we know from Taylor series that

$$\lim_{n\to\infty} c_n^{1/n} = \frac{1}{\text{radius of convergence of the } \textit{susceptibility } \sum_n c_n^{(\lambda)} z^n}.$$

So for z_n very near the boundary of the circle of convergence, i.e. $z_n \to 1/\mu$, the nth term of this Taylor series is $c_n^{(\lambda)} z_n^n \approx c_n/\mu^n$. In Part 3 of the talk, Ben will talk about choosing a sequence $z_n \to 1/\mu$ inductively with the aim of better understanding c_n/μ^n.

We will see that

$$\frac{c_n}{\mu^n} = A_\lambda\big[1 + O(n^{-\varepsilon})\big],$$

much finer scale behavior than just knowing the base of the exponential growth.

- We typically denote the *2-point function*

$$C_z^{(\lambda)}(x, y) := \sum_{n=0}^{\infty} c_n^{(\lambda)}(x, y) z^n,$$

and the *susceptability*

$$\chi^{(\lambda)}(z) = \sum_{x\in\mathbb{Z}^d} C_z^{(\lambda)}(0, x) = \sum_{n=0}^{\infty} \sum_{x\in\mathbb{Z}^d} C_z^{(\lambda)}(0, x) c_n^{(\lambda)}(x, y) z^n = \sum_n c_n^{(\lambda)} z^n$$

(the sum $\sum_x$ is actually a finite sum). If it makes any more sense to you, think of these as generating functions.

QUOTED FROM `https://personal.math.ubc.ca/~slade/sf.pdf#page=19`:

For $\lambda = 0$, we have seen in Section 1 that $c_n^{(0)} = |\Omega|^n$, and thus the number of n-step walks grows purely exponentially in n. There is overwhelming evidence to support the belief that for $\lambda \in (0, 1]$, the asymptotic form of $c_n^{(\lambda)}$ is given by

$$c_n^{(\lambda)} \sim A_\lambda \mu_\lambda^n n^{\gamma-1}.$$

Here, A_λ is a constant which, like μ_λ, depends on λ, d and Ω, but the critical exponent γ is independent of λ and Ω and is given by

$$\gamma = \begin{cases} 1 & \text{if } d = 1 \\ \frac{43}{32} & \text{if } d = 2 \\ 1.162\ldots & \text{if } d = 3 \\ 1 \text{ with logarithmic corrections} & \text{if } d = 4 \\ 1 & \text{if } d \geq 5 \end{cases}$$

The conjectured logarithmic correction in four dimensions, predicted by the renormalization group method, is

$$c_n^{(\lambda)} \sim A_\lambda \mu_\lambda^n (\log n)^{1/4} \quad \text{if } d = 4.$$

The independence of γ on $\lambda \in (0,1]$ and Ω is referred to as universality. Similarly, the power of the logarithm in (2.11) is believed to be universal.

2. Mean Squared Displacement

Start with another bombardment of formulas:

- By pure combinatorics because $c_n^{(0)}$ is a cardinality,

$$c_n^{(0)}(x) = \sum_{y\in\Omega} c_{n-1}^{(0)}(x-y) = \sum_{y\in\mathbb{Z}^d} \underset{c_1^{(0)}(y)}{\mathbb{1}_\Omega}\, c_{n-1}^{(0)}(x-y) = [c_1^{(0)}(\bullet) * c_{n-1}^{(0)}(\bullet)](x)$$

- Of course, upon seeing convolution, we look at the Fourier transform. Also, I should mention that another place where such convolutions pop up is exactly multiplying the generating functions of these quantities, namely precisely $C_z(x)$ defined earlier! Anyways, through the eyes of Fourier, the above convolution identity becomes

$$\widehat{c_n^{(0)}}(\xi) = (\widehat{c_1^{(0)}}(\xi))^n$$

- Defining the probability "D"ensity function

$$D(x) = \frac{1}{|\Omega|}\mathbb{1}_\Omega(x) \implies \widehat{D}(\xi) = \frac{1}{|\Omega|}\sum_{x\in\Omega} e^{i\xi x} = \frac{1}{2d}\sum_{i=1}^d e^{i\xi_i \cdot 1} + e^{i\xi_i(-1)} = \frac{1}{d}\sum_{i=1}^d \cos(\xi_i).$$

- Let us denote the variance of the p.d.f. $D(x)$ by σ^2. In the NN case, it equals 1. We have the following physical and Fourier space formulas:

$$\sigma^2 = \sum_{x\in\mathbb{Z}^d} |x|^2 D(x) = (i\nabla_\xi)^2 \widehat{D}(\xi)\Big|_{\xi=0} = -\Delta_\xi \frac{1}{|\Omega|}\widehat{c_1^{(0)}}(\xi)\Big|_{\xi=0}.$$

- Now recall that

$$\frac{c_n^{(0)}(x)}{c_n} := \text{probability that } n\text{-step SRW ends at } x = \mathbb{P}(\omega(n) = x)$$

where we can think of $\omega(n) = \sum_{i=1}^n \varepsilon_i$ for i.i.d. ε_i drawn from $D(x)$.

The CLT tells us that

$$\mathbb{P}\left(\frac{1}{\sigma\sqrt{n}}\omega(n)=x\right)=:\frac{c_n^{(0)}(x/\sigma\sqrt{n})}{c_n^{(0)}}$$

converges to the standard (multivariate) Gaussian p.d.f. on $\mathbb{R}^d$. In other words, if for every n-step path ω we color its endpoint a mostly transparent white, then looking at $\mathbb{Z}^d$ rescaled we get something that looks more and more like the height map of the standard Gaussian pdf.

- On the Fourier side, doing standard changes of variables produces

$$\frac{\widehat{c_n^{(0)}}(\xi/\sigma\sqrt{n})}{c_n^{(0)}}\to e^{-|\xi|^2/2d}.$$

Because $c_n^{(0)}$ happens to equal $\widehat{c_n^{(0)}}(0)$ (Fourier at 0 is integral of original over whole space), Slade in (1.15) writes this as

$$\frac{\widehat{c_n^{(0)}}(\xi/\sigma\sqrt{n})}{\widehat{c_n^{(0)}}(0)}\to e^{-|\xi|^2/2d}.$$

- Finally the *mean squared displacement* is defined as

$$\mathbb{E}_n^{(\lambda)}\left[\,|\omega(n)|^2\,\right]=\sum_{x\in\mathbb{Z}^d}|x|^2\frac{c_n^{(0)}(x)}{c_n^{(0)}}=\sum_{x\in\mathbb{Z}^d}|x|^2\frac{c_n^{(0)}(x)}{|\Omega|^n}=\sum_{x\in\mathbb{Z}^d}|x|^2(D)^{*n}$$

one can interpret as the "expected distance squared that a 'λ-level' self-repulsion (W)SAW gets".

The above integral of $|x|^2(D)^{*n}$ over the whole space is simply the Fourier transform evaluated at 0, so

$$\mathbb{E}_n^{(\lambda)}\left[\,|\omega(n)|^2\,\right]=-\Delta_\xi(\widehat{D})^n\Big|_{\xi=0},$$

which after some chian rule and product rule for $\nabla\cdot\nabla$ on some cosines, arrives at $=n\sigma^2$.

So for the NN-model where $\sigma^2=1$, we get that the "expected distance that the SRW gets" is $\sqrt{n}$, which basically matches the precise value given by Donsker's theorem and the Law of the Iterated Logarithm for Brownian Motion (those tell us that basically the walk will lie just about inside a circle of radius $\sqrt{n\log\log n}$). See below pictures of Slade.

QUOTED FROM `https://personal.math.ubc.ca/~slade/sf.pdf#page=20`:

The mean-square displacement is $\mathbb{E}_n^{(\lambda)}|\omega(n)|^2$ and it is believed that

$$\mathbb{E}_n^{(\lambda)}|\omega(n)|^2\sim v_\lambda n^{2\nu}$$

where v_λ is a constant depending on λ, d, Ω, and where ν is universal and given by

$$\nu = \begin{cases} 1 & \text{if } d = 1 \\ \frac{3}{4} & \text{if } d = 2 \\ 0.588\ldots & \text{if } d = 3 \\ \frac{1}{2} \text{ with logarithmic corrections} & \text{if } d = 4 \\ \frac{1}{2} & \text{if } d \geq 5. \end{cases}$$

The conjectured logarithmic correction to ν in four dimensions, predicted by the renormalization group, is

$$\mathbb{E}_n^{(\lambda)} |\omega(n)|^2 \sim v_\lambda n (\log n)^{1/4} \quad \text{if } d = 4$$

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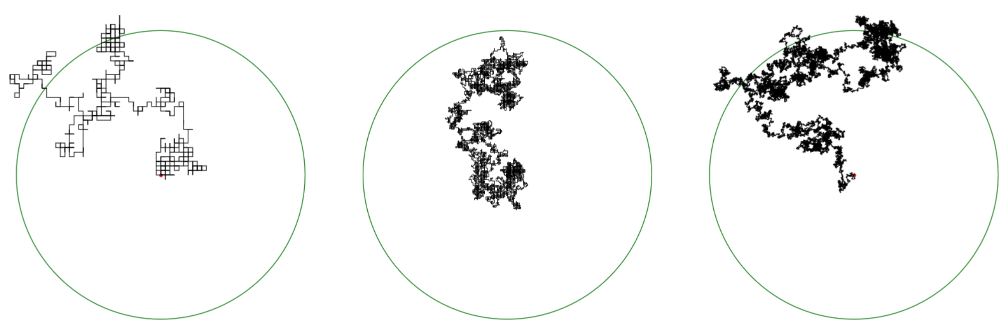

Fig. 1.1. Nearest-neighbour random walks on $\mathbb{Z}^2$ taking $n = 1,000$, 10,000 and 100,000 steps. The circles have radius $\sqrt{n}$, in units of the step size of the random walk.

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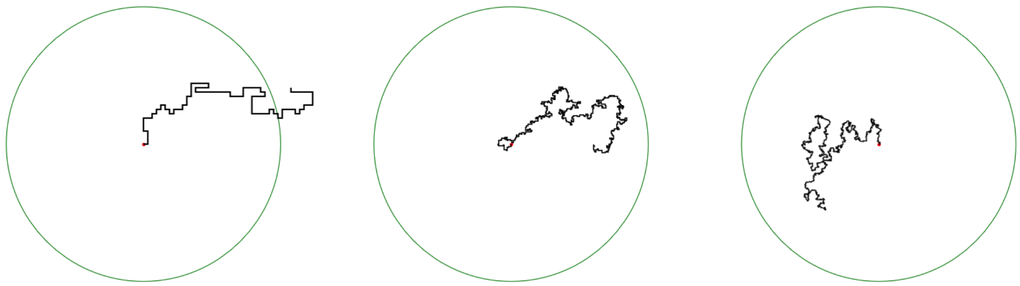

Fig. 2.1. Nearest-neighbour self-avoiding walks on $\mathbb{Z}^2$ taking $n = 100$, 1,000 and 10,000 steps, generated using the pivot algorithm [159]. The circles have radius $n^{3/4}$, in units of the step size of the self-avoiding walk.

3. Inclusion Exclusion

Allow me now to discuss a little of the naive inclusion-exclusion thinking that is later organized by the lace expansion method. Suppose we are counting purely SAW ($\lambda = 1$) starting at 0 and ending at x, i.e. we are trying to calculate the quantity $c_n^{(1)}(x)$.

Here's the first approximation: every SAW $0 \to x$ is of the form a length 1 SAW from $0 \to y$ for $y \in \Omega$, and a length $n-1$ SAW from $y \to x$. Things of this form (FORM0: length 1 SAW $0 \to y$ and independent SAW $y \to x$ of length $n-1$) are counted by

$$\text{Approx} := [c_1^1 * c_{n-1}^1](x).$$

The issue is that 2 SAWs $0 \to y$ and $y \to x$ chained together may NOT produce a SAW from $0 \to x$: the (only) issue that can happen is that the SAW $y \to x$ may touch 0!

So, the exceptions are of the form: a SAW loop starting at ending at 0 ("SAW loop" meaning the only intersection is at the endpoints) of length m, and a SAW starting at 0 of length $n-m$. Things of this form (FORM1: SAW loop length m starting and ending at 0 and independent SAW starting at 0 going to x of length $n-m$) are counted by

$$\text{Correction1} := \sum_{m=1}^{n} [\pi_m^{[1]}(0) \cdot c_{n-m}^1(x)]$$

where $\pi_m^{[1]}(0)$ denotes the number of length m SAW loops starting and ending at 0. By defining $\pi_m^{[1]}(x)$ to be the number of walks consisting of exactly 1 loop, where 0 is the start and end of a loop and x is the start and end of a loop, we get $\pi_m^{[1]}(x) = \pi_m^{[1]}(0) \cdot 1_{x=0}$, and we can rewrite the above

$$\text{Correction1} = \sum_{m=1}^{n} [\pi_m^{[1]}(x) * c_{n-m}^1](x)$$

So we have $c_n^1(x) \approx \text{Approx} - \text{Correction1}$.

But we took away too much! Because we treated the two pieces of FORM1 as independent, we ignored all the interactions between the SAW loop counted by $\pi_m^{[1]}(0)$ and the SAW counted by $c_{n-m}^1(x)$! We must now ADD BACK things of the form: SAW loops L starting and ending at 0, and SAW walk W starting at 0, but the loop L and walk W intersect! Draw picture of laces, with bridge between 0 and m, and also bridges between $m^- \in \{0, \dots, m\}$ and $m^+ > m$, or even perhaps more bridges between $m^{--} \in \{0, \dots, m\}$ and $m^{++} > m$.

We end up considering quantities like $\pi_m^{[2]}(x) :=$ the number of walks consisting of exactly 2 loops, where 0 is the start and end of a loop and x is the start and end of a loop, and convolutions of these quantities.

Drawing these lace diagrams, we cut up the walk into pieces that are self-avoiding, which we then count by pretending they are independent. Then to correct for this pretending, we consider finer and finer self-intersections, thus building up an expression involving lower and lower terms. Carson will now organize these ideas using the formalism of lace expansions!

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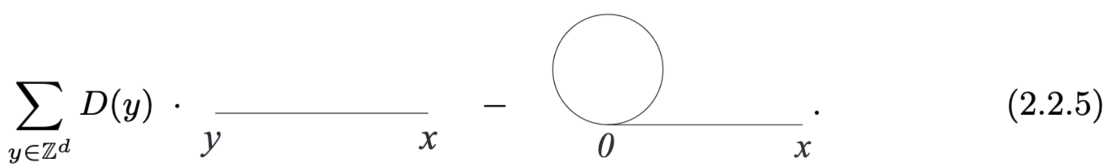

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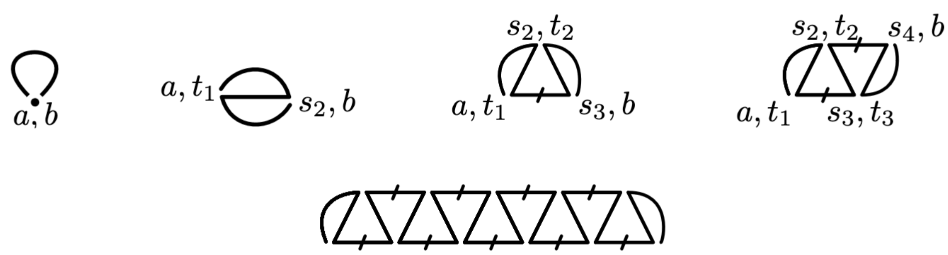

Fig. 3.4. Self-intersections required for a walk ω with $\prod_{st \in L} U_{st}(\omega) \neq 0$, with U_{st} given by (2.1), for the laces with $N = 1, 2, 3, 4$ bonds depicted in Fig. 3.2. The picture for $N = 11$ is also shown.

In this section we define some of the relevant quantities pertaining to a simple random walk on $\mathbb{Z}^d$; we will analyze the same quantities for the self-avoiding walk (hereafter SAW). We will use the lace expansion to estimate them.

3.1. **Simple Random Walk as Nearest Neighbor Walk.** We begin by defining the nearest-neighbor random walk. Denote

$$\Omega = \{x \in \mathbb{Z}^d : \|x\|_1 = 1\}.$$

Then a map $\gamma : \{0, \dots, n\}$ is a nearest-neighbor walk if $\gamma(i) - \gamma(i-1) \in \Omega$ for $1 \le i \le n$ (heuristically, the steps are 'chosen independently' from all other steps in the model).

Denote by $\mathcal{W}_n(x, y)$ the set of walks γ starting at x, taking n steps, and ending at y. We put $\mathcal{W}_0(x, y) = \delta_{x,y}$. Then write $c_n(x, y) = |\mathcal{W}_n(x, y)|$.

We demand the model to be translation-invariant, in the sense that $\mathcal{W}_n(0, y-x) = \mathcal{W}_n(x, y)$. Going forward, we will assume all walks start at the origin and denote $\mathcal{W}_n(y - x) := \mathcal{W}_n(x, y)$, so that $c_n(y - x) := c_n(0, y - x) = c_n(x, y)$. The number $c_n(x)$ is the number of nearest-neighbor walks of length n ending at x (which is 0 if $n < \|x\|_1$). We also define $c_n = \sum_{x \in \mathbb{Z}^d} c_n(x)$ as the number of nearest-neighbor walks starting at the origin that end anywhere

By simple counting, we observe that

$$c_n(x) = \sum_{y \in \mathbb{Z}^d} c_{n-1}(x - y) = \sum_{y \in \mathbb{Z}^d} c_1(y) c_{n-1}(x - y), \text{ i.e. } c_n(x) = (c_1 * c_{n-1})(x),$$

where on the right we mean the discrete convolution of the two functions.

This suggests the use of Fourier analysis to obtain bounds on the coefficients c_n. We define the (discrete) Fourier transform of an absolutely summable function $f(x) : \mathbb{Z}^d \to \mathbb{C}$ by

$$\hat{f}(\xi) := \sum_{x \in \mathbb{Z}^d} f(x) e^{i\xi \cdot x}, \quad \xi \in [-\pi, \pi)^d,$$

with inverse

$$\int_{[-\pi,\pi)^d} \frac{1}{(2\pi)^d} \hat{f}(\xi) e^{-i\xi \cdot x} \, d\xi.$$

Another useful identity is, of course, that $c_n = \hat{c}_n(0)$. By the usual properties of the Fourier transform, we get

$$\hat{c}_n(\xi) = \hat{c}_1(\xi) \hat{c}_{n-1}(\xi) \implies \hat{c}_n(\xi) = (\hat{c}_1(\xi))^n.$$

The characteristic function of the nearest neighbor walk, equivalently the transition probability, is

$$D(x) = \frac{1}{|\Omega|} \mathbb{1}_{\{x \in \Omega\}} = \frac{1}{|\Omega|} c_1(x) \implies \hat{D}(\xi) = \frac{1}{|\Omega|} \hat{c}_1(\xi).$$

This means that, for $\xi = (\xi_1, \dots, \xi_d) \in [-\pi, \pi)^d$,

$$\hat{c}_n(k) = |\Omega|^n \, (\hat{D}(\xi))^n = |\Omega|^n \left(\frac{1}{d} \sum_{j=1}^d \cos(\xi_j) \right)^n.$$

The two-point function $C(x)$ is defined as

$$C_z(x) = \sum_{n\geq 0} \sum_{\gamma \in \mathcal{W}_n(x)} z^n = \sum_{n\geq 0} c_n(x) z^n.$$

Think $0 < z < 1$ as the probability with which you independently choose each edge in the walk; then this is the number of walks that can reach x of any length, as long as neighboring edges are chosen independently with probability z.

Fourier transforming, we get

$$\hat{C}_z(\xi) = \sum_{x\in\mathbb{Z}^d} \hat{c}_n(\xi) z^n = \frac{1}{1 - z\,|\Omega|\,\hat{D}(\xi)}.$$

The *susceptibility* $\chi(z) = \hat{C}_z(0) = (1 - z\,|\Omega|)^{-1}$, intuitively a measure of the degree to which the nearest-neighbor walks can 'permeate' the lattice (since $\hat{C}(0) = \sum_{x\in\mathbb{Z}^d} C(x)$). It has a singularity at $z_c := |\Omega|^{-1}$, which is called the *critical point*).

Finally, another quantity of interest will be the *mean-square displacement* of a SRW starting at the origin. We begin by computing the variance of the characteristic function D. By Plancherel, one has

$$\sigma^2 = \sum_{x\in\mathbb{Z}^d} |x|^2\, D(x) = -\Delta_\xi \hat{D}(\xi)\big|_{\xi=0}.$$

By the central limit theorem and that $\hat{c}_n(\xi) = |\Omega|^n\, \hat{D}(\xi)^n$, we have

$$\lim_{n\to\infty} \frac{\hat{c}_n(\xi/\sigma\sqrt{n})}{\hat{c}_n(0)} = e^{-|k|^2/2d},$$

and the mean-square displacement

$$\frac{\sum_{x\in\mathbb{Z}^d} |x|^2\, c_n(x)}{\sum_{x\in\mathbb{Z}^d} c_n(x)} = -\Delta \hat{D}(\xi)^n\big|_{\xi=0} = n\sigma^2.$$

We will be able to prove similar results for the (W)SAW using lace expansion. There, the variance σ^2 in this last formula will roughly be replaced by the so-called diffusion coefficient, which we will also estimate as part of our argument.

3.2. **Self-Avoiding Walk.** The SAW is a nearest-neighbor walk γ on $\{0,\ldots,n\}$ such that $\gamma(s) \neq \gamma(t)$ for any $0 \leq s \neq t \leq n$.

The SAW is a model that was 'invented' by Flory, a chemist, as a model for determining the spatial arrangement of macromolecules such as polymers, which tend to organize in chains: the bonds between the molecules are roughly the same length, and (obviously) two molecules cannot occupy the same physical site in space. The molecules are the vertices on $\mathbb{Z}^d$, and the edges of a nearest-neighbor walk are the chemical bonds connecting them. Include pictures during presentation!

Though we will without apology use the same notation and terminology for SAWs, we should emphasize that the SAW is non-Markovian, in the sense that we can't choose the steps at a given lattice site independently of all other steps. (Part of the point of the lace expansion is to perform computational techniques to regain some of this independence at the cost of accruing some error terms that we hope to be able to control).

The generalization of this model counts walks which are not only self-avoiding but only *weakly* self-avoiding, in the following sense: write st for an edge $\{s, t\}$ in γ (that is, there is and i for which $\gamma(i) = s$ and $\gamma(i+1) = t$ or vice versa) for $0 < \lambda < 1$, we define

$$c_n^{(\lambda)}(x) = \sum_{\gamma \in \mathcal{W}_n(x)} \prod_{st \in \gamma} \left(1 - \lambda \mathbb{1}_{\{\gamma(s)=\gamma(t)\}}\right).$$

Thus, walks which intersect themselves are penalized by a weight λ. When $\lambda = 1$, the walk is strictly self-avoiding; when $\lambda = 0$, we count all nearest-neighbor walks.

It's also possible (and does not make the argument more involved) to let λ depend on the specific edge:

$$c_n^{(\lambda)}(x) = \sum_{\gamma \in \mathcal{W}_n(x)} \prod_{st \in \gamma} \left(1 - \lambda_{st} \mathbb{1}_{\{\gamma(s)=\gamma(t)\}}\right).$$

In the sequel, we will write $U_{st} = -\lambda_{st} \mathbb{1}_{\gamma(s)=\gamma(t)}$.

We define $c_n^{(\lambda)}$ and the two-point function $C_z^{(\lambda)}$ analogously to the SRW. The radius of convergence of $C_z^{(\lambda)}$ is $\lim_n (c_n^{(\lambda)})^{1/n}$. This limit exists because $c_n^{(\lambda)}$ is a subadditive sequence: $c_{n+m}^{(\lambda)} \leq c_n^{(\lambda)} c_m^{(\lambda)}$. When $\lambda = 1$ (i.e. when the walk is strictly self-avoiding), the limit is called the *connective constant* μ.

Empirical evidence suggests that

$$c_n^{(\lambda)} \sim A_\lambda \mu_\lambda^n n^{\gamma - 1},$$

where A_λ and μ_λ depend on d and λ, but γ is not (the asymptotic behavior motivates some of the inductive hypotheses we'll see later – this part can go in section 3).

4. The Lace Expansion

We define

$$c_n(x) := \sum_{\omega \in \mathcal{W}_n(x)} K[0,n](\omega) \tag{1}$$

$$K[a,b](\omega) := \sum_{a \le s < t \le b} (1 + U_{st}(\omega)) \tag{2}$$

In short, K is the weight attached to a path ω, and $c_n(x)$ is cumulative weight of all n step paths from 0 to x. The U_{st} should be understood as negative weights that penalize self-intersection; typically, $U_{st} = -\lambda_{st} \mathbb{1}_{\omega(s)=\omega(t)}$ for some $\lambda_{st} \in [0,1]$. Taking $\lambda \equiv 0$ reduces to a simple random walk, while taking $\lambda \equiv 1$ reduces to a strictly self-avoiding walk.

Our goal in this section will be to decompose $c_n(x)$ and $K[a,b](\omega)$ in a such a way as to enable estimates inductive in the time variable n.

4.1. **Pairings.**

Let $I = [a,b] \cap \mathbb{Z}$ be an interval of integers. We define a pairing over I, written $\{s_1t_1, \ldots, s_Nt_N\}$, to be a set of pairs with $a \le s_l < t_l \le b$. A pairing covers I if for every $i \in (a,b) \cap I$, there is a pair s_lt_l with $s_l < i < t_l$. Write $\mathcal{P}[a,b]$, $\mathcal{C}[a,b]$, for the set of pairings and covering pairings over I respectively.

We remark that this terminology is nonstandard; in the literature pairings are called "graphs" and covering pairings are called "connected graphs". We adopt different terminology to avoid confusion over the interpretation of connectedness in our context.

We will use these notions to decompose the path weight function $K[a,b](\omega)$. First, by expanding the product we have

$$K[a,b](\omega) := \prod_{a \le s < t \le b} (1 + U_{st}(\omega)) = \sum_{\Gamma \in \mathcal{P}[a,b]} \prod_{st \in \Gamma} U_{st}(\omega) \tag{3}$$

We can partition pairings over $[a,b]$ as follows: if the pairing has an edge with left endpoint a, then it necessarily covers some maximal interval $[a,j]$, and by maximality it has no edges with left endpoint less than j and right endpoint greater than j. In other words, if we write $\mathcal{C} \cdot \mathcal{P}$ for the set of pairings formed by concatening a pairing from the left with a pairing from the right, we have obtained that

$$\mathcal{P}[a,b] = \mathcal{P}[a+1,b] \sqcup \bigsqcup_{a < j \le b} \mathcal{C}[a,j] \cdot \mathcal{P}[j,b] \tag{4}$$

Applying this to (3) and factoring, we get directly that

$$K[a,b](\omega) = K[a+1,b](\omega) + \sum_{j=a+1}^{b} J[a,j](\omega) K[j,b](\omega) \text{ where} \tag{5}$$

$$J[a,b](\omega) := \sum_{\Gamma \in \mathcal{C}[a,b]} \prod_{st \in \Gamma} U_{st}(\omega) \tag{6}$$

Returning to $c_n(x)$, we get

$$\begin{aligned} c_n(x) &= \sum_{\omega \in \mathcal{W}_n(x)} K[0,n](\omega) \\ &= \sum_{\omega \in \mathcal{W}_n(x)} \left(K[1,n](\omega) + \sum_{m=1}^{n} J[0,m](\omega) K[m,n](\omega) \right) \end{aligned}$$

The $K[1,n]$ terms can be interpreted as giving a $c_{n-1}(\cdot)$ for each possible first step in the walk:

$$\sum_{\omega \in \mathcal{W}_n(x)} K[1,n] = \sum_{\omega_1 \in \Omega} \sum_{\omega' \in \mathcal{W}_{n-1}(x-\omega_1)} K[0,n-1](\omega') = \sum_{\omega_1 \in \Omega} c_{n-1}(x-\omega_1) \tag{7}$$

Since a path of length 1 has no self-interaction, $c_1(y)$ is just the indicator that y is reachable from 0, so we conclude that

$$\sum_{\omega \in \mathcal{W}_n(x)} K[1,n](\omega) = (c_1 * c_n)(x) \tag{8}$$

By similarly splitting along ω_m, we have

$$\sum_{\omega \in \mathcal{W}_n(x)} J[0,m](\omega) K[m,n](\omega) = \sum_{y \in \mathbb{Z}^d} \sum_{\omega' \in \mathcal{W}_m(y)} J[0,m](\omega') c_{n-m}(x-y) \tag{9}$$

Substituting, we obtain the recursive expansion:

$$c_n(x) = (*c_{n-1})(x) + \sum_{m=1}^{n} (\pi_m * c_{n-m})(x) \tag{10}$$

$$\pi_m(y) := \sum_{\omega' \in \mathcal{W}_m(y)} J[0,m](\omega') \tag{11}$$

The conceptual theme of the past few steps has been to reduce the computation of $c_n(x)$ to that of c_{n-1} with a simple random walk step prepended (forming the first term) plus corrections (forming the second term) which only need to account for paths ω which have a loop at 0.

(10) is often referred to as the lace expansion in the literature, yet we have not seen any laces! In fact, the role of laces is to further decompose π into sum whose terms count specific types of self-intersections.

4.2. **Laces.** We define a lace $L \in \mathcal{C}[a,b]$ to be a covering pairing which is minimal under set inclusion; that is, if any pair $s_l t_l$ is removed from L, it is no longer covering. We write $\mathcal{L}[a,b]$ for the set of laces, and $\mathcal{L}^{(N)}[a,b]$ for the set of laces consisting of N pairs.

Given an arbitrary $\Gamma \in \mathcal{C}[a,b]$, we associate the lace L_Γ obtained by the greedy algorithm, taking $s_1 = a, t_1 = \max\{t : at \in \Gamma\}$, and subsequently $t_{i+1} = \max\{t : \exists s < t_i, st \in \Gamma\}, s_{i+1} = \min\{s : st_{i+1} \in \Gamma\}$, terminating with $t_{i+1} = b$.

Now that we have a map from covering pairings to laces, we can rewrite the sum over covering pairings in the definition of J as a sum over laces. We get:

$$\begin{aligned} J[a,b](\omega) := & \sum_{\Gamma\in\mathcal{C}[a,b]} \prod_{st\in\Gamma} U_{st}(\omega) \\ = & \sum_{L\in\mathcal{L}[a,b]} \sum_{\Gamma\in\mathcal{C}[a,b], \mathsf{L}_\Gamma = L} \prod_{st\in L} \prod_{s't'\in\Gamma\setminus L} U_{st}(\omega) U_{s't'}(\omega) \\ = & \sum_{L\in\mathcal{L}[a,b]} \prod_{st\in L} U_{st}(\omega) \sum_{\Gamma\in\mathcal{C}[a,b], \mathsf{L}_\Gamma = L} \prod_{s't'\in\Gamma\setminus L} U_{s't'}(\omega) \end{aligned}$$

We say an pair $st \notin L$ is compatible with L if $\mathsf{L}_{L\cup\{st\}} = L$, and write $\mathsf{C}(L)$ for the set of pairs compatible with L. In light of the greedy algorithm, compatibility can be understood as an ordering property for s and t relative to the $s_1t_1, \ldots, s_ml_m$ that make up L. An equivalent formulation is: if $t \in (t_j, t_{j+1})$, then $s \geq t_j$, otherwise if $t = t_j$, then $s > s_j$. With this, it is clear that adding any collection of compatible pairs to L leaves the result of the greedy algorithm unchanged, and conversely, $\mathsf{L}_\Gamma = L$ precisely when $\Gamma \setminus L$ consists of pairs compatible with L.

Using the notion of compatibility, we can further transform the portion of the sum corresponding to $\Gamma \setminus L$:

$$\sum_{\Gamma\in\mathcal{C}[a,b], \mathsf{L}_\Gamma = L} \prod_{s't'\in\Gamma\setminus L} U_{s't'}(\omega) = \sum_{\Gamma'\subseteq\mathsf{C}(L)} \prod_{s't'\in\Gamma'} U_{s't'}(\omega) = \prod_{s't'\in\mathsf{C}(L)} (1 + U_{s't'}(\omega)) \tag{12}$$

Additionally splitting the sum over laces by number of pairs, we conclude that

$$J[a,b](\omega) = \sum_{N=1}^{\infty} J^{(N)}[a,b](\omega) \text{ where} \tag{13}$$

$$J^{(N)}[a,b](\omega) := \sum_{L\in\mathcal{L}^{(N)}[a,b]} \prod_{st\in L} U_{st}(\omega) \prod_{s't'\in\mathsf{C}(L)} (1 + U_{s't'}(\omega)) \tag{14}$$

and analogously,

$$\pi_m(y) = \sum_{N=1}^{\infty} \pi_m^{(N)}(y) \text{ where} \tag{15}$$

$$\pi_m^{(N)}(y) := \sum_{\omega\in\mathcal{W}_m(y)} J^{(N)}[0,m] \tag{16}$$

This is the form of the lace expansion used for estimating π_m. Since the U_{st} are nonpositive, $\operatorname{sign} J^N[a,b] = \operatorname{sign} \pi_m^{(N)} = (-1)^N$. Some sources opt to pull out the sign so that the $\pi_m^{(N)}$ themselves are nonnegative. We also remark that the series defining J and π_m are in fact finite sums, since the size of a lace is bounded in terms of the size of the interval it covers.

4.3. **Motivation for the Lace Expansion.** Recall that the total weight of a path ω is the product

$$\prod_{0\leq s<t\leq n} (1 + U_{st}(\omega))$$

Recalling that $U_{st} = 0$ if $\omega(s) \neq \omega(t)$, and $U_{st} \approx -1$ if $\omega(s) = \omega(t)$, we see that the term corresponding to a lace L,

$$\prod_{st \in L} U_{st}(\omega) \prod_{s't' \in \mathsf{C}(L)} (1 + U_{s't'}(\omega))$$

counts contributions of factors corresponding to self-intersections at the time-pairs of the lace, and non-self-interactions at the time-pairs compatible with the lace.

5. Main Results

As an application of lace expansion techniques, der Hofstad, den Hollander, and Slade (1998) obtained Gaussian asymptotics for weakly self-avoiding random walks with an exponential penalty. Specifically, for parameters p, β, we define the weight function

$$c_n(x) = \sum_{\omega \in \mathcal{W}_n(x)} \exp\left(-\beta \sum_{0 \le s < t \le n} \mathbb{1}_{\omega(s)=\omega(t)} |t-s|^{-p}\right) \tag{17}$$

Then their result can be stated as:

Theorem 1. *Fix $\varepsilon = p + (d-4)/2$. If $d > 4$, we require only that $p \ge 0$. If $d \le 4$, we require that $p > (4-d)/2$;. Then there are constants $A, \mathcal{D}, \mu$ depending on d, p, β so that*

$$c_n = A\mu^n(1 + O(n^{-\varepsilon})) \tag{18}$$

$$\frac{1}{c_n} \sum_{x \in \mathbb{Z}^d} |x|^2 c_n(x) = \begin{cases} \mathcal{D}n(1 + O(n^{-1\wedge\varepsilon}) & \varepsilon \ne 1 \\ \mathcal{D}n(1 + O(n^{-1}\log n) & \varepsilon = 1 \end{cases} \tag{19}$$

$$\frac{\hat{c}_n(\xi/\sqrt{\mathcal{D}n})}{\hat{c}_n(0)} = \exp(-|\xi|^2/2d(1 + O(n^{-\delta'})), \tag{20}$$

where $0 < \delta' < \min(1, \varepsilon/2)$ is arbitrary and the estimate in (20) is uniform in ξ provided $|\xi|^2/\log n$ is sufficiently small.

Theorem 2. *Assume either that $d \ge 4$ and $p > 0$ (note the change!) or $d \le 4$ and $p > (4-d)/2$,$\beta_0 = \beta_0(d,p) > 0$ such that for $\beta < \beta_0$,*

$$\frac{c_n(x)}{c_n} = 2\left(\frac{d}{2\pi D}\right)^{d/2} n^{-d/2} e^{-d|x|^2/2Dn}(1 + o_{n\to\infty}(1)), \tag{21}$$

provided that n has the same parity as $\|x\|_1$, and the estimate is uniform in x provided $|x|^2/(n\log n)$ is sufficiently small.

If $d \ge 4$ but $p = 0$, we only have the weaker estimate

$$\sup_{x\in\mathbb{Z}^d} \frac{c_n(x)}{c_n} = O(n^{-d/2}).$$

5.1. **The (Six!) Inductive Hypotheses.** Now we spell out the setup of the inductive argument, leaving most of the details out. It should be noted that the induction argument, though technical, is largely mechanical. Once one knows what hypotheses are appropriate, their verification requires no technology beyond simple estimates. This gives the inductive approach some advantage over other techniques which make use of the lace expansion by relating it to other specialized tools such as cluster expansion to estimate the coefficients π_m, c_n.

Another advantage of the inductive argument is that it gives good asymptotics which indicate how the mean-squared displacement etc. approach the Gaussian behavior at finite scales n, giving quantitative information about the convergence.

Recall the Fourier transform of the two point-function $C_z^{(\lambda)}(x)$ is given by

$$\hat{C}_z^{(\lambda)}(\xi) = \sum_{n\geq 0} \hat{c}_n^{(\lambda)}(\xi) z^n.$$

We want bounds on $c_n^{(\lambda)} = \sum_{x\in\mathbb{Z}^d} c_n^{(\lambda)}(x) = \hat{c}_n^{(\lambda)}(0)$. To this end, denote

$$A_n(\xi) = z^n \hat{c}_n^{(\lambda)}(\xi);$$

we will be interested, of course, in the case $\xi = 0$. Later, we will suppress the λ-s from the notation to make the presentation easier to read.

Observe that $c_n(x)$ is nonzero if and only if n and $\|x\|_1$ have the same parity, and by the properties of the Fourier transform, $A_n(\xi + \pi\vec{e}_1) = (-1)^n A_n(\xi)$. Thus it will suffice to consider $\xi \in [-\pi/2, \pi/2] \times [-\pi, \pi)^{d-1}$.

The susceptibility $\chi^{(\lambda)}(z) = \sum_{n\geq 0} c_n^{(\lambda)} z^n$ has radius of convergence $1/\mu_\lambda$. In the inductive scheme, we will estimate the terms in the sum with the values of z being constrained to smaller and smaller intervals converging to this critical value. This is the only place where we can get information about c_n: converging to any smaller value of z will land us within the radius of convergence, so the limiting value of the terms in the series will be zero, while converging to larger values will cause the terms to diverge.

5.2. **Consequences of the induction.** We will not advance the induction, rather, we will take the conclusions of the argument for granted. Here, we will explain how the induction leads to the proof of the theorem.

Lemma 3. *Assume (H1-H4) and (H6). Then*

(1) $\sup_{1\leq j\leq n} |A_j(0)| \leq \exp(C\beta)$ *where* C *is independent of* β *and* n*;*

(2) $\|A_j\|_1 \lesssim j^{-d/2}$ *for* $1 \leq j \leq n$*;*

(3) $|\Delta A_j(0)| \lesssim j$ *for* $1 \leq j \leq n$.

We remark that (1) follows immediately from (H3) with setting $\xi = 0$, while (3) is an immediate consequence of (H2), (H3), and (H6). The proof of (2) takes a bit more work and uses some of the technical overhead we have decided to suppress in our exposition, so we skip it.

Note that I_1 is bounded away from 0 if β is sufficiently small, so $1/z \leq C$ holds uniformly for $z \in \bigcap_{1\leq j\leq n} I_j$.

By Fourier inversion, we see that $\|c_j\|_\infty = \frac{1}{(2\pi)^d} \|A_j\|_1$. As promised, gaining control of the $\|A_j\|_1$ gives us control over the lace expansion coefficients $\hat{\pi}_m$:

Lemma 4. *There is a constant* C_1 *depending on* d, p *but not on* β *such that for all* n*, for* $2 \leq m \leq n+1$*,* $z \in I_n$*, and* $\xi \in [-\pi, \pi)^d$*, we have*

(1) $|\hat{\pi}_m(\xi)| z^m \leq C_1 \beta m^{-2-\varepsilon}$,

(2) $|\Delta \hat{\pi}_m(\xi)| z^m \leq C_1 \beta^2 m^{-1-\varepsilon}$*, and*

(3) $\left|\hat{\pi}_m(\xi) - \hat{\pi}_m(0) - [1 - \hat{D}(\xi)]\Delta\hat{\pi}_m(0)\right| z^m \leq C_1\beta^2 \left|\xi\right|^{2+2\eta} m^{-1-\varepsilon-2\eta}$,

where $0 < \eta < 1$, and the value of C_1 can be deduced from Lemma 3.

5.3. **Identifying the Constants.** Our calculations also tell us what the constants in the preceding theorems are:

Theorem 5.

$$1 = 2d\mu^{-1} + \sum_{m=2}^{\infty} \hat{\pi}_m(0)\mu^{-m} \tag{22}$$

$$A = \left[2d\mu^{-1} + \sum_{m=2}^{\infty} m\hat{\pi}_m(0)\mu^{-m}\right]^{-1} \tag{23}$$

$$D = A\left[2d\mu^{-1} \sum_{m=2}^{\infty} \Delta\hat{\pi}_m(0)\mu^{-m}\right] \tag{24}$$