Things you should definitely know by memory (to the point where you can recite it): FILL IN THE BLANKS (I've filled in some of them for you)

- When people write the words "let V be a vector space", they implicitly mean 4 pieces of data:

  1. a set V, whose elements we call ________
  2. a base field F, whose elements we call ________
  3. an operation $+$, called ____________, which eats 2 vectors $\in V$ as input, and spits out one output
  4. an operation $\cdot$, called ______________, which eats one ______ $\in$ ___ AND one ______ $\in$ ___ as input, and spits out one output

  that satisfy the following 10 axioms: FILL IN THE BLANKS (I've filled in some of them for you)

	$+$	$\cdot$
Closure	The image of the $+$ function lies in $\subseteq V$, i.e. all outputs that $+$ spits out are vectors in V. Or more mathematically, $\forall v_1, v_2 \in V$, we want $v_1 + v_2 \in V$.	The image of the $\cdot$ function lies in $\subseteq V$, i.e. all outputs that $\cdot$ spits out are vectors in V. Or more mathematically, $\forall a \in F, v \in V$, we want $a \cdot v \in V$.
Commutativity	$\forall v_1, v_2$, we want $v_1 + v_2 = v_2 + v_1$	
Associativity		
Identity		Let 1_F denote the (unique) multiplicative identity in the field F. Then $\forall v \in V$, we want $1_F \cdot v = v$.
Inverse		

Distributatativity (2 axioms for VS)

- For a set $S \subseteq V$, a **linear combination (LC) over S** or **linear combination (LC) of elements of S** is a FINITE sum of the form $\sum_{i \in I}$ ______ with **coefficients** $a_i \in$ ___, and $v_i \in$ ___ for ALL $i \in I$, where I is some FINITE set of indices.

  Or if you don't want to talk about index sets I, you can say instead: a **LC over S** is a FINITE sum of the form $\sum_{i \in 1}^{n}$ ______ with $a_i \in$ ___, $v_i \in$ ___ for ALL $i = 1, \ldots, n$, where n is some natural number $n \in \mathbb{N} := \{0, 1, 2, \ldots, \}$

- A set $\mathscr{L}$ of vectors in V is **linearly independent** if:

$$\forall n \in \mathbb{N}, \quad \forall a_1, \ldots, a_n \in F, \quad \forall v_1, \ldots, v_n \in \mathscr{L},$$

$$\textbf{if} \text{ the linear combination } \sum_{i=1}^{n} a_i v_i = 0, \ \textbf{then} \text{ all the } a_1, \ldots, a_n \text{ must have be } 0.$$

In English, this would be "the only linear combination over $\mathscr{L}$ that equals 0, is the trivial linear combination (all the coefficients are 0)"

  - to PROVE a set $S \subseteq V$ is LI, you should begin with the phrase: "To show that S is LI, let us consider an arbitrary linear combination over S that equals 0, i.e. $\sum_{i=1}^{n} a_i v_i = 0$ with $v_i \in S, a_i \in F$. We want to show that the coefficients a_i (for $i = 1, \ldots, n$) are all $= 0$."
  - if you want to USE the known fact that a set $S \subseteq V$ is LI, it would look like this: suppose you are at a point in your proof where you have a linear combination $\sum_{i=1}^{n} b_i v_i = 0$, with $v_i \in S$, which you know equals 0; then you can say "Because S is LI, and we have that the linear combination $\sum_{i=1}^{n} b_i v_i = 0$, we can now conclude the coefficients of this linear combination, b_i (for $i = 1, \ldots, n$), are all $= 0$."
  - Here's an example where I use both the above phrases. **CLAIM:** if $\{u, v\} \subseteq V$ is a LI set, then so is $\{u, u + v\}$.

    Proof: "To show that $\{u, u + v\}$ is LI, let us consider an arbitrary linear combination over $\{u, v + v\}$ that equals 0, i.e. $a_1 u + a_2(u + v) = 0$ with $a_1, a_2 \in F$. We want to show that the coefficients a_1, a_2 are both $= 0$."

    Observe that $a_1 u + a_2(u + v) = 0$ can be rearranged to $(a_1 + a_2)u + a_2 v = 0$ (using the various distributativity, commutativity, and associativity properties of $+$ and $\cdot$).

    "Because $\{u, v\}$ is LI, and we have that $(a_1 + a_2)u + a_2 v = 0$, we can now conclude that the coefficients of this linear combination, $(a_1 + a_2), a_2$, are both $= 0$."

    Finally, now that we know $a_2 = 0$ and $a_1 + a_2 = 0$, we conclude that also $a_1 = 0$. So we have shown that both coefficients $a_1, a_2 = 0$, and we are done.

EXERCISE: prove that $\{u, v, w\}$ LI $\implies$ $\{u, u + v, u + v + w\}$ LI, using the above language!

- A set $\mathscr{D}$ of vectors in V is **linearly dependent** if:

$$\exists n \in \mathbb{N}, \quad \exists a_1, \ldots, a_n \in F, \quad \exists v_1, \ldots, v_n \in \mathscr{D}, \text{ such that}$$

$$\text{not all the } a_1, \ldots, a_n \text{ equal } 0 \text{ (i.e. at least 1 is } \neq 0\text{), but STILL the LC } \sum_{i=1}^{n} a_i v_i = 0$$

In English, this would be "there is a non-trivial linear combination over $\mathscr{D}$ that equals 0"

  - to PROVE a set $S \subseteq V$ is LD, you should (somehow) cook up a non-trivial linear combination over $\mathscr{D}$ that equals 0, and say "We have produced some NON-zero $a_1, \ldots, a_n \in F$ and $v_1, \ldots, v_n \in \mathscr{D}$ s.t. $\sum_{i=1}^{n} a_i v_i = 0$, and so by definition, S is LD."
  - if you want to USE the known fact that a set $S \subseteq V$ is LD, it would look like this: "Because S is LD, by definition there must exists some natural number n, and NON-zero $a_1, \ldots, a_n \in F$ and $v_1, \ldots, v_n \in \mathscr{D}$ such that $\sum_{i=1}^{n} a_i v_i = 0$."

EXERCISE: given $S_1 \subseteq S_2$, prove (use direct proof!) that if S_1 is LD, then S_2 is LD also. Use the above language!

- The **Cutting Down/Culling Theorem:** $\forall$ (finite) spanning lists $\mathcal{S}$, by reading from head to tail and kicking out any vectors that are "redundant", i.e. in the span of the vectors preceding it in the list $\mathcal{S}$, I can cut it down to an LI list (that is still spanning).

  To USE the Culling Theorem, you would say "We can use the Culling Theorem on the (finite) spanning set $\mathcal{S}$, which tells us we can cut it down to a subset $\mathcal{S}' \subseteq \mathcal{S}$ that is still spanning, but now guaranteed to be LI. (i.e. $\mathcal{S}'$ is a basis)"

- The **Replacement Theorem:** $\forall$ LI lists $\mathscr{L}$, and $\forall$ spanning lists $\mathcal{S}$, I can replace elements of $\mathcal{S}$ by elements of $\mathscr{L}$ one-by-one (so the list "$\mathcal{S}$" is updating/changing at every step!!!), (by inserting each element of $\mathscr{L}$ at the head of $\mathcal{S}$, and kicking the first element (reading head to tail) of $\mathcal{S}$ that is "redundant", i.e. in the span of the vectors preceding it in the list $\mathcal{S}$), so that $\mathcal{S}$ remains spanning (at every step). The end result is a new spanning list $\mathcal{S}'$ of the same size as the original $\mathcal{S}_{\text{original}}$, but now with $\mathscr{L}$ contained within it: $\mathscr{L} \subseteq \mathcal{S}'$.

  To USE the Replacement Theorem, you would say "Because $\mathscr{L}$ is LI and $\mathcal{S}$ is spanning, we can apply the Replacement Theorem, which produces a new spanning set $\mathcal{S}'$ (of the same size as $\mathcal{S}$), but now contains $\mathscr{L}$."

  Using these 2 theorems, you should be able to prove ALL of the following corollaries: (You should do ALL these proofs, to master using the Replacement Theorem. I did the first one to show you. You may want to come back to these after doing some of the later problems on this document.)

  - **Corollary 1:** every LI set has a smaller size than every spanning set. i.e. $\forall \mathscr{L}$ LI and $\forall \mathcal{S}$ spanning, we have $|\mathscr{L}| \leq |\mathcal{S}|$.

    Proof: let us fix (like "fix in place", not "repair") $\mathscr{L}$ an arbitrary LI set, and $\mathcal{S}$ an arbitrary spanning set. We want to show that $|\mathscr{L}| \leq |\mathcal{S}|$.

    "Because $\mathscr{L}$ is LI and $\mathcal{S}$ is spanning, we can apply the Replacement Theorem, which produces a new spanning set $\mathcal{S}'$ (of the same size as $\mathcal{S}$), but now contains $\mathscr{L}$."

    So, we have $\mathscr{L} \subseteq \mathcal{S}' \implies |\mathscr{L}| \leq |\mathcal{S}'| = |\mathcal{S}|$, and we are done.

  - **Corollary 2:** if $\mathscr{B}_1, \mathscr{B}_2$ are 2 bases of the F-VS V, they are the same size. In other words, all bases of a given vector space V are the same size. This size is called the dimension of V, denoted $\dim_F V$.

    Proof: (Hint: use Cor. 1). ____

  - **Corollary 3:** suppose V is a finite-dimensional F-VS. If $W \subseteq V$ is a subspace, show that $m := \dim_F W \leq \dim_F V =: n$. Also show that if $\dim_F W = \dim_F V$, then in fact $W = V$.

    Proof: I'll start. W has dimension m means there is a basis $\{w_1, \ldots, w_m\}$ of W. And V having dimension n means there is a basis $\{v_1, \ldots, v_n\}$ of V. (Hint: use Replacement Theorem). ____

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– **Corollary 4:** suppose V is a finite-dimensional F-VS, with $\dim_F V =: n$. Let $\mathscr{L}$ be an LI set. Prove that $|\mathscr{L}| \le n$ (intuitively, "LI sets must be small"). Also prove that if $|\mathscr{L}| = n$, then $\mathscr{L}$ is also spanning (and hence automatically a basis).

Proof: (Hint: for the first part, use the Replacement theorem on $\mathscr{L}$ and any basis of V. For the second part, use Cor. 3, with $W = \operatorname{span}_F \mathscr{L}$). ______________________

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– **Corollary 5:** suppose V is a finite-dimensional F-VS, with $\dim_F V =: n$. Let $\mathscr{S}$ be a spanning set. Prove that $|\mathscr{S}| \ge n$ (intuitively, "spanning sets must be big"). Also prove that if $|\mathscr{S}| = n$, then $\mathscr{S}$ is also LI (and hence automatically a basis).

Proof: (Hint: use the Culling Theorem.) ______________________

I'll give you starting point for the 2nd part: suppose we are given some spanning set $\mathscr{S}$ with $|\mathscr{S}| = n$.

Suppose f.s.o.c. that it's not LI. So it's LD. "We can use the Culling Theorem on the (finite) spanning set $\mathscr{S}$, which tells us we can cut it down to a subset $\mathscr{S}' \subseteq \mathscr{S}$ that is still spanning, but now guaranteed to be LI. i.e. $\mathscr{S}'$ is a basis.".

Because $\mathscr{S}'$ is LI, and $\mathscr{S}$ we said was LD, and $\mathscr{S}' \subseteq \mathscr{S}$, it must be that they are not the same, or in other words, $\mathscr{S}'$ is a STRICT subset $\subsetneq \mathscr{S}$. So $|\mathscr{S}'| < |\mathscr{S}| = n$. (Hint: finish by using Cor. 2) ______________________

– **Corollary 6:** suppose V is a finite-dimensional F-VS. Let $\mathscr{L}$ be an LI set, and $\mathscr{B}$ be a basis of V. Prove that applying the Replacement Theorem to infiltrate $\mathscr{L}$ into $\mathscr{B}$, resulting in (a still spanning set) $\mathscr{B}'$ (of the same size as $\mathscr{B}$), is so that $\mathscr{B}'$ also remains LI (and is hence still a basis).

Proof: (Hint: use Cor. 5.) ______________________

1. Recall that $P(\mathbb{F}) := \{f(x) = a_k x^k + \cdots a_1 x + a_0 \mid a_i \in \mathbb{F}, k = 0, 1, 2, 3, \ldots\}$ is the vector space of polynomials in $\mathbb{F}$. What is implicit base field for this VS (vector space)? What are the implicit addition, and scalar multiplication operations to make the set $P(\mathbb{F})$ a VS (over the base field you just named)?

2. What is the span of the set
$$S = \{1, 1 + x, 1 + x + x^2\}$$
in $P(\mathbb{F})$?

3. Find a subspace $U \subseteq P(\mathbb{F})$ such that
$$S = \{1, 1 + x, 1 + x + x^2\}$$
is a **basis** of U. (You must prove that S is a basis)

Let $U = \{\langle x_1, x_2, x_3, x_4, x_5\rangle \in \mathbb{R}^5 \mid x_1 = 3x_2, \text{ and } x_3 = 7x_4\}$ be a subspace of $\mathbb{R}^5$.

1. Find a basis $\mathscr{B}$ of U.

2. Extend your basis from part (a) to a basis of $\mathbb{R}^5$, by using the Replacement theorem/algorithm on the LI list $\mathscr{B}$ and the spanning list $\{(1, 0, 0, 0, 0), \ldots, (0, 0, 0, 0, 1)\}$ (the standard basis for $\mathbb{R}^5$)?

Consider the vector space $\mathbb{R}^3$ over $\mathbb{R}$. The following vectors generate $\mathbb{R}^3$:

$$v_1 = (0, 2, -1)\,, \quad v_2 = (3, 2, 1)\,, \quad v_3 = (3, 6, -1)\,, \quad v_4 = (2, 0, -3) \quad \text{and} \quad v_5 = (1, 2, 3)\,.$$

1. Apply the Cutting Down theorem/algorithm to the vectors v_1, v_2, v_3, v_4, v_5 to construct a basis for $\mathbb{R}^3$.

2. Apply the Cutting Down theorem/algorithm to the vectors v_3, v_5, v_2, v_4, v_1 to construct a basis for $\mathbb{R}^3$.

3. Are your answers in (a) and (b) the same or different? Why?

1. Let V be a vector space over a field $\mathbb{F}$ and W_1, W_2 subspaces of V.

   (a) Show that $W_1 \cup W_2$ need not be a subspace of V.

   (b) The **sum** of W_1 and W_2 is defined by

   $$W_1 + W_2 := \{w_1 + w_2 \mid w_1 \in W_1 \text{ and } w_2 \in W_2\}$$

   Prove that $W_1 + W_2$ is a subspace of V.

   (c) Prove that the span of $W_1 \cup W_2$ is exactly $W_1 + W_2$. (Remember to prove equality of sets $A = B$, you must show both directions: $A \subseteq B$, i.e. letting $x \in A$ be arbitrary, show that also $x \in B$; and also the other way around: $B \subseteq A$, i.e. letting $y \in B$ be arbitrary, show that also $y \in A$.)

2. On the next page is a proof given for the statement: The set of vectors $\langle 1, 0, 0\rangle, \langle 1, 1, 0\rangle, \langle 1, 1, 1\rangle \in \mathbb{R}^3$ is linearly independent.

   Explain the mistake(s) in the proof.

   Let $a, b, c \in \mathbb{R}$. Suppose $a\langle 1, 0, 0\rangle + b\langle 1, 1, 0\rangle + c\langle 1, 1, 1\rangle = 0$. If $a = b = c = 0$, then $a\langle 1, 0, 0\rangle + b\langle 1, 1, 0\rangle + c\langle 1, 1, 1\rangle = \langle a + b + c, b + c, c\rangle = \langle 0, 0, 0\rangle$. Therefore, the set of vectors $\langle 1, 0, 0\rangle, \langle 1, 1, 0\rangle, \langle 1, 1, 1\rangle$ is linearly independent.

3. Let V be a vector space over a field $\mathbb{F}$ and $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$ linearly independent vectors. Do the following steps to show that $\boldsymbol{u}+\boldsymbol{v}, \boldsymbol{v}+\boldsymbol{w}, \boldsymbol{w}$ are linearly independent.

(a) State what it means for $\boldsymbol{u}+\boldsymbol{v}, \boldsymbol{v}+\boldsymbol{w}, \boldsymbol{w}$ to be linearly independent.

(b) Express 0 as a linear combination of $\boldsymbol{u}+\boldsymbol{v}, \boldsymbol{v}+\boldsymbol{w}, \boldsymbol{w}$.

(c) Rewrite the linear combination of (b) in terms of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.

(d) Use that $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ is linearly independent to obtain a system of linear equations.

(e) Solve the system of linear equations, and conclude that $\boldsymbol{u}+\boldsymbol{v}, \boldsymbol{v}+\boldsymbol{w}, \boldsymbol{w}$ are linearly independent.

Let V be a vector space and let W_1 and W_2 be finite-dimensional subspaces of V.

1. Show that $W_1 + W_2$ is finite-dimensional (**Hint:** prove this without finding an explicit basis for $W_1 + W_2$).

2. Suppose that $\{v_1, \ldots, v_m\}$ is a basis of W_1, and suppose that $\{w_1, \ldots, w_n\}$ is a basis of W_2. If $W_1 \cap W_2 = \{0\}$, find a basis for $W_1 + W_2$.

   You must prove your set is a basis.

3. Prove or give a counterexample to the following statement:

   Let V be a vector space and let W_1 and W_2 be finite-dimensional subspaces of V. Then $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2)$.